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Exponential stability and stabilization of sampled-data systems with a time-varying period

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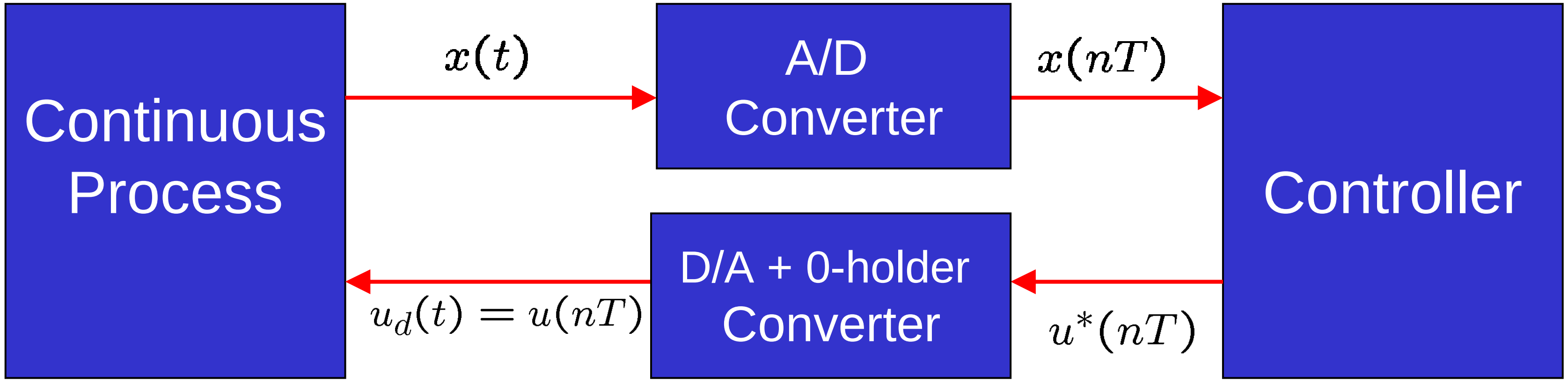
Motivation : Stability of sampled-data systems

Hybrid problem : both *continuous and discrete-time problem*



In a realistic case, $$u_d(t) = u(nT(t))$$

Problems to solve:

•***To ensure the stability of the process;***

•***Provide less convervative conditions to obtain an accurate upper-bound of T***

Application to the stabilization problem

Consider a linear system of the form:

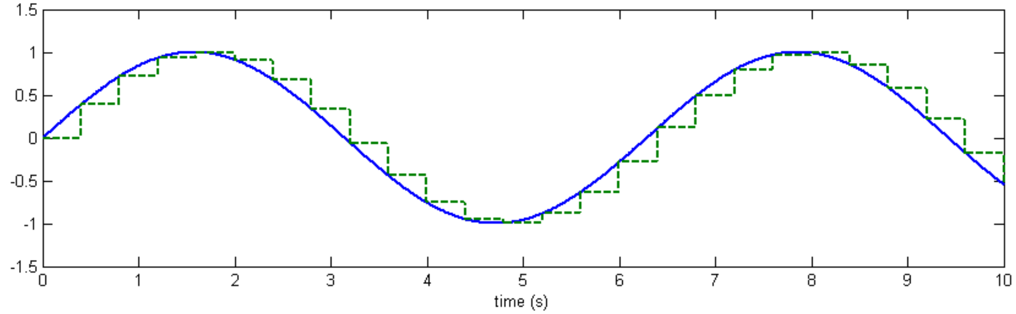
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t_k), \quad t_k \leq t < t_{k+1}$$

A and ***B*** can be constant, uncertain,…

What kind of sampling?

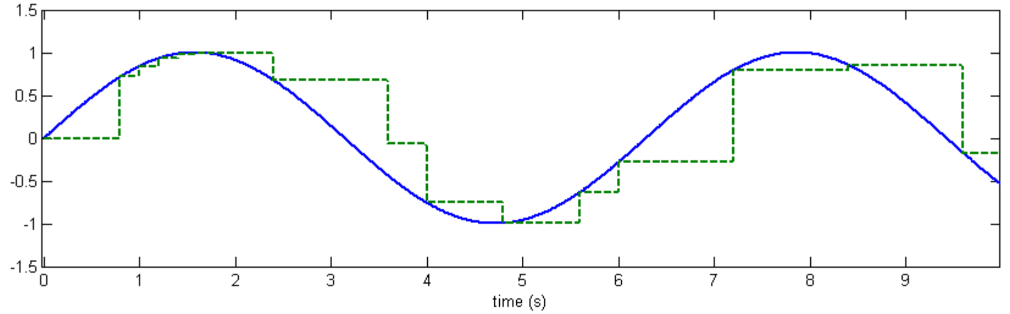
- **Periodic**

$$t_{k+1} - t_k = T$$



- **Aperiodic (asynchronous)**

$$t_{k+1} - t_k \neq T$$
$$t_{k+1} - t_k \leq \bar{T}$$



find $\bar{T}$???

Outline

1. ***A time-delay modelling for sampled input***

2. ***Review of existing results:***
   - Time-delay approach for periodic and aperiodic samplings
   - Small gain and robust approaches
   - Impulsive system approach

3. ***Asymptotic Stability***

4. ***Exponential Stability and stabilization***

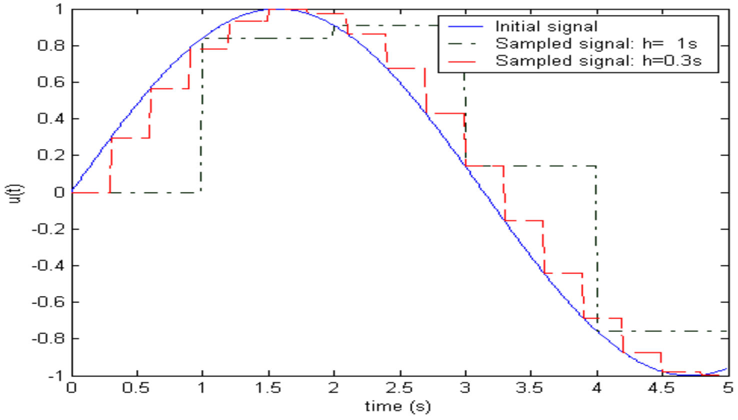
5. ***Examples***

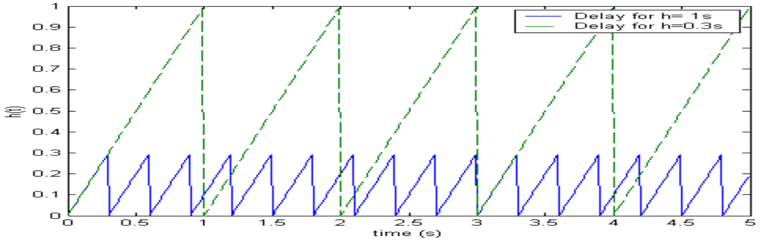
6. ***Conclusion***

A) An input delay approach

Interesting idea…

Mikheev et al. 88, Sobolev et al. 89. Aström et al. 92 .





Sampled-data signal

(here with uniform sampling)

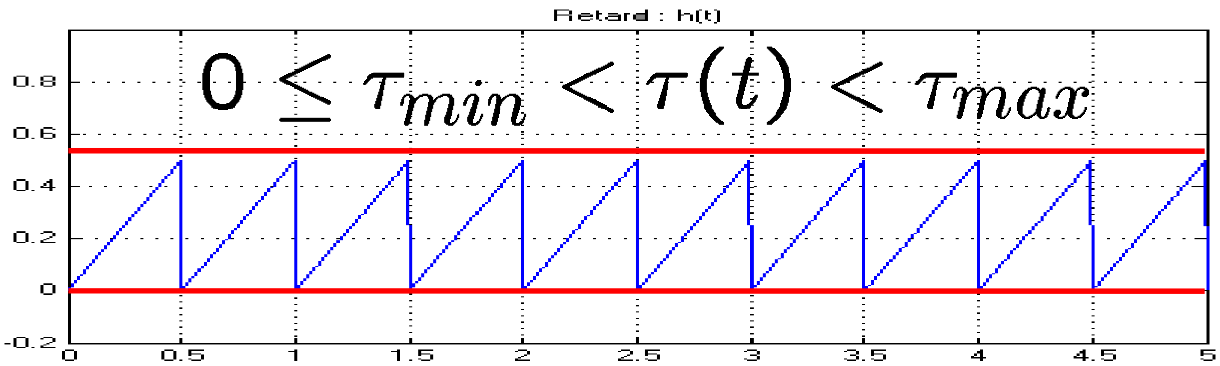
⇕

Delayed signal with

$$\tau(t) = t - t_k$$
$$\dot{\tau}(t) = 1$$

$$u(t) = u_d(t_k) = u_d(t - [t - t_k]) = u(t - \tau(t))$$

A) An input delay approach



Dependence on the bounds of the sampling period (possible to consider input delay)

Special case for TDS : $\dot{\tau}(t) = 1$

Th 1: [Fridman et al, Auto. 2004]

For a given gain K, the closed-loop system is asymptotically stabl, for all samplings whose period is less than τ_{max}, if there exist $n \times n$ matrices $0 < P_1$, P_2, P_3, Z_1, Z_2, Z_3 and $R > 0$ that satisfy:

$$\Psi_1 < 0, \quad \text{and} \quad \begin{bmatrix} R & [0\ K^T B^T] P \\ * & Z \end{bmatrix} \geq 0,$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix}, \quad \Psi_1 = \Psi_0 + \tau_2 Z + \begin{bmatrix} 0 & 0 \\ 0 & \tau_2 R \end{bmatrix},$$

$$\Psi_0 = P^T \begin{bmatrix} 0 & I_n \\ A + BK & -I_n \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ A + BK & -I_n \end{bmatrix}^T P.$$

$$\left(V(t) = x^T(t) P_1 x(t) + \int_{-\tau_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \bar{x}(s) ds d\theta\right)$$

Motivations:

1. Holds for both periodic and aperiodic samplings and also additional input delay

2. Linear wrt. the system parameters A and B (TV systems, param. unc. and sat.)

but what about the conservatism??

A) An input delay approach

Consider the following example:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(t_k)$$

Stability under periodic samplings	T_{max}
Theoretical bounds	1.729
Theorem 1 (aperiodic)	0.8696

- **Considerable conservatism**
- **Take into account aperiodic sampling and « non linearities » (saturation)**

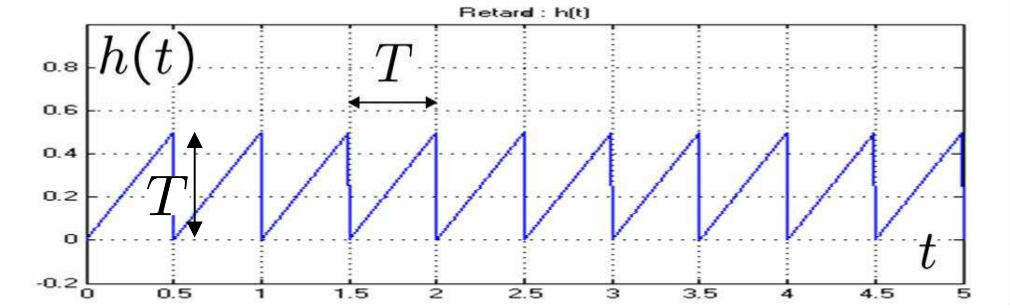
- **Where does it come from?**

Theorem 1 is not only dedicated to sampled-data systems but also to systems with any input delays (i.e. constant delay)

1) An input delay approach

$$\dot{x}(t) = -ax(t) - bx(t - h(t)) \qquad (1)$$

$$h(t) = t - kT \text{ pour } kT < t \le (k+1)T$$

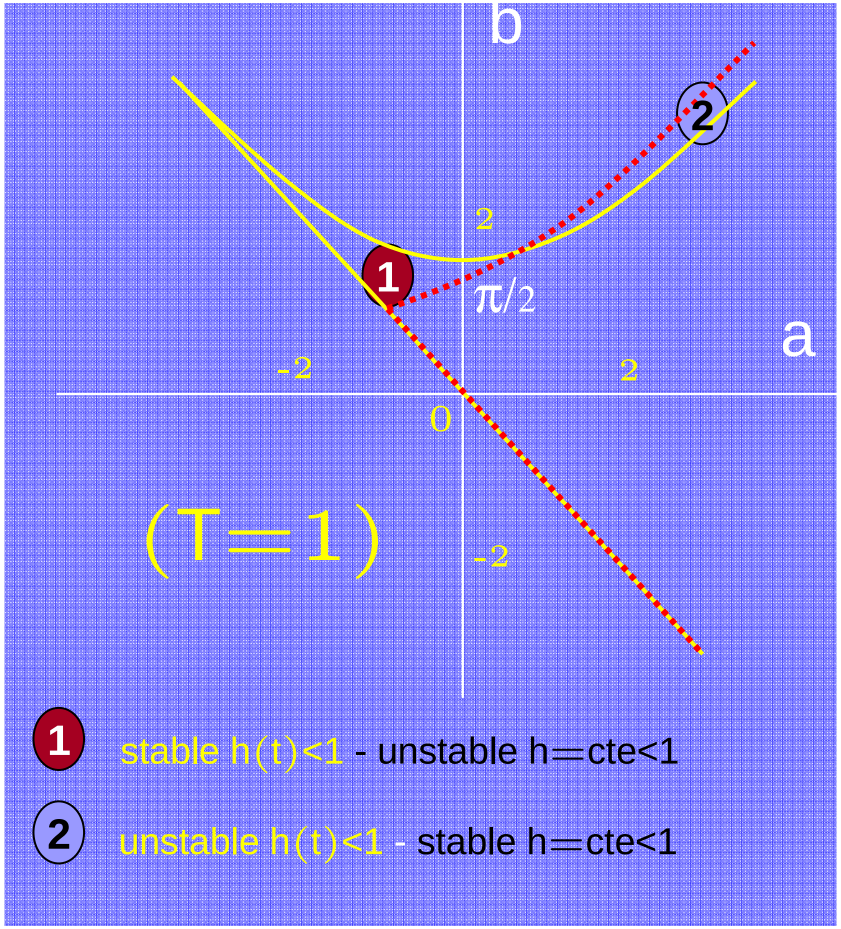


is asymptotically stable iff (yellow area)

$$\left|(1+\tfrac{b}{a})e^{-aT} - \tfrac{b}{a}\right| < 1 \quad \text{si } a \neq 0$$

$$|1 - bT| < 1 \quad \text{si } a = 0$$

and, for h = cste ∈ [0,1] iff (red area)



No equivalence between delay and sampling

Requires more accurate analysis

2) Small gain theorem approaches

Stability under aperiodic samplings	T_{max}
Theorem 1	0.869
Mirkin, IEEE TAC 2007 (small gain theorem)	1.365
Fujioka, Automatica, 2009 (Th.2) (Small gain theorem)	1.635

2. Review of existing papers

3) Impulse systems appraoch

Improvement of the input delay approach

Th 2: [Nagahshtabrizi, SCL. 2008]

The system is asymptotically stable if there exist P, R, X_1 positive definite matrices and a matrice N that satisfy:

$$M_1 + \tau_{max} M_2 < 0 \text{ and } \begin{bmatrix} M_1 & \tau_{max} N \\ * & -\tau_{max} R \end{bmatrix}$$

where

$$\bar{F} = [A \quad BK];$$
$$M_1 = 2[P \ 0]^T \bar{F} - [I \ \ -I]^T (X_1 [I \ \ -I] + 2X_2 [0 \ I]) - 2N[I \ \ -I] + \tau_{max} \bar{F}^T R \bar{F}$$
$$M_2 = 2\bar{F}^T (X_1 [I \ \ -I] + X_2 [0 \ I])$$

$$V'(t) = x^T(t)Px(t) + \int_{t_k}^{t} (\tau_{max} - t + s)\dot{x}^T(s)R\dot{x}(s)ds$$
$$+(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_1 (x(t) - x(t_k))$$
$$+2(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_2 x(t_k)$$

⇨ $(\dot{\tau}(t) = 1)$

Stability conditions	Tmax
Theoretical bounds	1.729
Theorem 1 (periodic or not)	0.8696
Theorem 2 (aperiodic) $(X_2 = 0)$	1.1137
Theorem 2 (periodic)	1.3277

} **Better but still conservative**

3. Main result

Definition of a more appropriate piecewise continuous LKF

$$V'(t) = x^T(t)Px(t) + \int_{t_k}^{t} (\tau_{max} - t + s)\dot{x}^T(s)R\dot{x}(s)ds$$
$$+(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_1 (x(t) - x(t_k))$$
$$+2(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_2 x(t_k)$$

We introduce t_k

$$V'(t) = x^T(t)Px(t) + \int_{t_k}^{t} (\underbrace{(\tau_{max} - (t - t_k))}_{\tau_{max} - \tau(t)} + \underbrace{(s - t_k)}_{\tau(s)})\dot{x}^T(s)R\dot{x}(s)ds$$
$$+(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_1 (x(t) - x(t_k))$$
$$+2(\tau_{max} - \tau(t))(x(t) - x(t_k))^T X_2 x(t_k)$$

$$V'(t) = V(t) + \int_{t_k}^{t} \tau(s)\dot{x}^T(s)R\dot{x}(s)ds \rightarrow V'(t) \geq V(t)$$
$$\dot{V}'(t) = \dot{V}(t) + \tau(t)\dot{x}^T(t)R\dot{x}(t) \rightarrow \dot{V}'(t) \geq \dot{V}(t)$$

Theorem 3

Assume that there exist symmetric positive definite matrices P, R and $S_1 \in R^{n\times n}$ and two matrices $S_2 \in R^{n\times n}$ and $N \in R^{2n\times n}$ that satisfy

$$\Pi_1 + T\Pi_2 < 0, \qquad \begin{bmatrix} \Pi_1 & TN \\ * & -TR \end{bmatrix} < 0,$$

where

$$\Pi_1 = 2M_1^T P M_0 - M_3^T S_1 M_3 - 2M_2^T S_2 M_3 - 2NM_3,$$
$$\Pi_2 = M_0^T(RM_0 + 2S_1M_3 + 2S_2M_2)$$

and $M_0 = \begin{bmatrix} A & A_d \end{bmatrix}$, $M_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & I \end{bmatrix}$, $M_3 = \begin{bmatrix} I & -I \end{bmatrix}$. The system is thus asymptotically stable for a constant period T.

Extensions:

i) ***Aperiodic*** sampling periods : Using a linearity argument wrt. T

ii) Polytopic type of ***uncertainties*** (Constant or T-V)

Same type of LKF:

$$V(t) = x^T(t)Px(t) + (T-\tau(t))\int_{t_k}^{t} \dot{x}^T(s)R\dot{x}(s)ds$$
$$+(T-\tau(t))(x(t)-x(t_k))^T(S_1(x(t)-x(t_k))$$
$$+2(T-\tau(t))(x(t)-x(t_k))^T S_2 x(t_k)$$

Stability conditions	τ_m
Theoretical bounds	1.729
Theorem 1 (periodic or not)	0.8696
Theorem 2 (aperiodic $(X_2 = 0)$ & periodic)	1.1137 & 1.3277
Theorem 3 (aperiodic & periodic) (similar to [Fridman, Auto.10])	1.719

Proof of Theorem 3:

Consider
$$V(t) = x^T(t)Px(t) + (T-\tau(t))\int_{t_k}^{t}\dot{x}^T(s)R\dot{x}(s)ds$$
$$+(T-\tau(t))(x(t)-x(t_k))^T S_1(x(t)-x(t_k))$$
$$+2(T-\tau(t))(x(t)-x(t_k))^T S_2 x(t_k)$$

V is ***continuous*** for periodic samplings and ***discontinuous but decreasing*** for aperiodic samplings. The differentiation leads to

$$\left(\xi(t) = \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}\right)$$

$$\dot{V}(t) = \xi^T(t)[2M_1PM_0 - M_3^T S_1 M_3 - 2M_3^T S_2 M_2$$
$$+(T-\tau(t))M_0^T(RM_0 + 2S_1M_3 + 2S_2M_2]\xi(t)$$
$$-\int_{t_k}^{t}\dot{x}^T(s)R\dot{x}(s)ds$$

Noting that for all ***N***: $-\int_{t_k}^{t}\dot{x}^T(s)R\dot{x}(s)ds \leq \xi^T(t)[-2NM_3 + \tau(t)NR^{-1}N^T]\xi(t),$

We obtain $\forall \tau(t) \in [0\ T], \quad \dot{V}(t,x_t) \leq \xi^T(t)[\Pi(\tau(t))]\xi(t)$

where $\Pi(\tau(t)) = \Pi_1 + (T-\tau(t))\Pi_2 + \tau(t)(NR^{-1}N^T - \Pi_2)$

As $\Pi(\tau(t))$ is ***linear*** with respect to $\tau(t)$, the conditions $\Pi(0) < 0$ and $\Pi(T) < 0$ ensure that $\forall \tau(t) \in [0\ T], \quad \dot{V}(t,x_t) \leq 0.$

Theorem 4

For a given $\alpha \in R/\{0\}$, assume that there exist symmetric positive definite matrices P, R and $S_1 \in R^{n\times n}$ and two matrices $S_2 \in R^{n\times n}$ and $N \in R^{2n\times n}$ that satisfy

$$\Pi_1 + f_\alpha(T,0)\Pi_2 < 0, \qquad \begin{bmatrix} \Pi_1 & g_\alpha(T,T)N \\ * & -g_\alpha(T,T)R \end{bmatrix} < 0,$$

where

$$\Pi_1 = 2M_1^T P M_0 - M_3^T S_1 M_3 - 2M_2^T S_2 M_3 - 2NM_3 + 2\alpha M_1^T P M_1,$$
$$\Pi_2 = M_0^T (RM_0 + 2S_1 M_3 + 2S_2 M_2)$$

and the same matrices M_0, M_1, M_2 and M_3. The functions f_α and g_α are

$$\begin{array}{ll} \text{if } \alpha \neq 0, & f_\alpha(T,\tau) = (e^{2\alpha(T-\tau)} - 1)/2\alpha, \\ \text{if } \alpha > 0, & g_\alpha(T,\tau) = e^{2\alpha T}(1 - e^{-2\alpha\tau})/2\alpha, \\ \text{if } \alpha < 0, & g_\alpha(T,\tau) = (1 - e^{-2\alpha\tau})/2\alpha, \end{array} \qquad (1)$$

The system is thus exponentially stable with an exponential decay rate α for a constant period T.

Extensions:

i) ***Aperiodic*** sampling periods: By a linearity argument wrt. T

ii) α can be positive or negative, allowing estimating the ***divergence rate***

Sketch of the proof:

Ensure that $\dot{V}(t) + 2\alpha V(t) < 0$ $\quad V(t) \le e^{-2\alpha(t-t_0)} V(t_0)$

With an appropriate function
$$\begin{cases} f_\alpha(T,\tau) = (e^{2\alpha(T-\tau)} - 1)/2\alpha \xrightarrow[\alpha \to 0]{} f_0(T,\tau) = (T-\tau) \\ \dot{f}_\alpha(T,\tau) + 2\alpha f_\alpha(T,\tau) = 1 \end{cases}$$

Additionnal conditions to the LMI variables
Based on the same functional and the same LMI

$$\begin{aligned} V(t) = & x^T(t)Px(t) + (T-\tau(t))\int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds \\ & +(T-\tau(t))(x(t)-x(t_k))^T S_1(x(t)-x(t_k)) \\ & +2(T-\tau(t))(x(t)-x(t_k))^T S_2 x(t_k) \end{aligned}$$

⇒

$$\Pi_1 + f_\alpha(T,0)\Pi_2 < 0, \quad \begin{bmatrix} \Pi_1 & g_\alpha(T,T)N \\ * & -g_\alpha(T,T)R \end{bmatrix} < 0,$$

Theorem4 $P > 0,\ R > 0\ S_1 > 0 \quad \forall S_2$ ⇒ $V(t) > 0$

Theorem5 $\Pi_3 = \begin{bmatrix} P + f_\alpha(T,\tau)S_1 & f_\alpha(T,\tau)(S_2 - S_1) \\ f_\alpha(T,\tau)(S_2^T - S_1) & f_\alpha(T,\tau)(S_1 - S_2 - S_2^T) \end{bmatrix} > 0, \quad R > 0$

Similar to [Fridman, Automatica 2010]

$$V(t) = \xi^T(t)\Pi_3\xi(t) + f_\alpha(T,\tau)\int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds > 0$$

Theorem6 $\Pi_4 = \begin{bmatrix} P + f_\alpha(T,\tau)(S_1 + R/T) & f_\alpha(T,\tau)(S_2 - S_1 - R/T) \\ f_\alpha(T,\tau)(S_2^T - S_1 - R/T) & f_\alpha(T,\tau)(S_1 - S_2 - S_2^T + R/T) \end{bmatrix} > 0,$

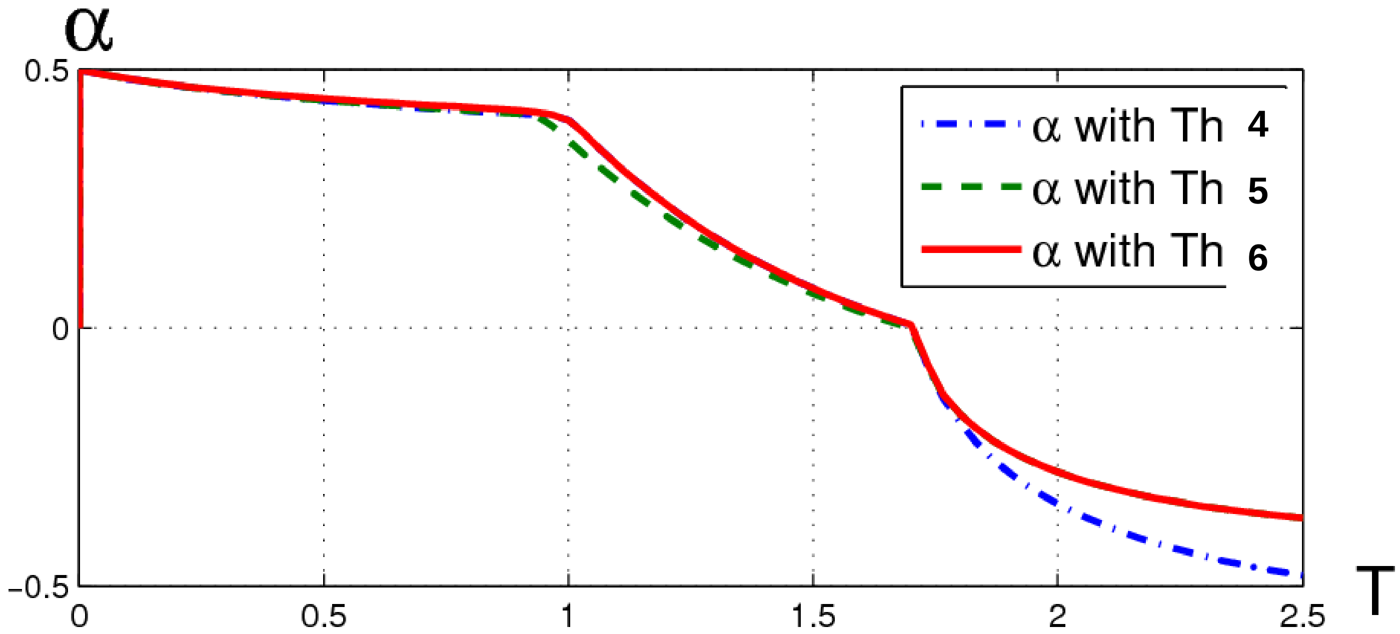
Similar to [Fridman, Automatica 2010]

$$V(t) = \xi^T(t)\Pi_3\xi(t) + f_\alpha(T,\tau)\int_{t_k}^t \dot{x}^T(s)R\dot{x}(s)ds \geq \xi^T(t)\Pi_4\xi(t) > 0$$

4. Examples

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(t_k)$$

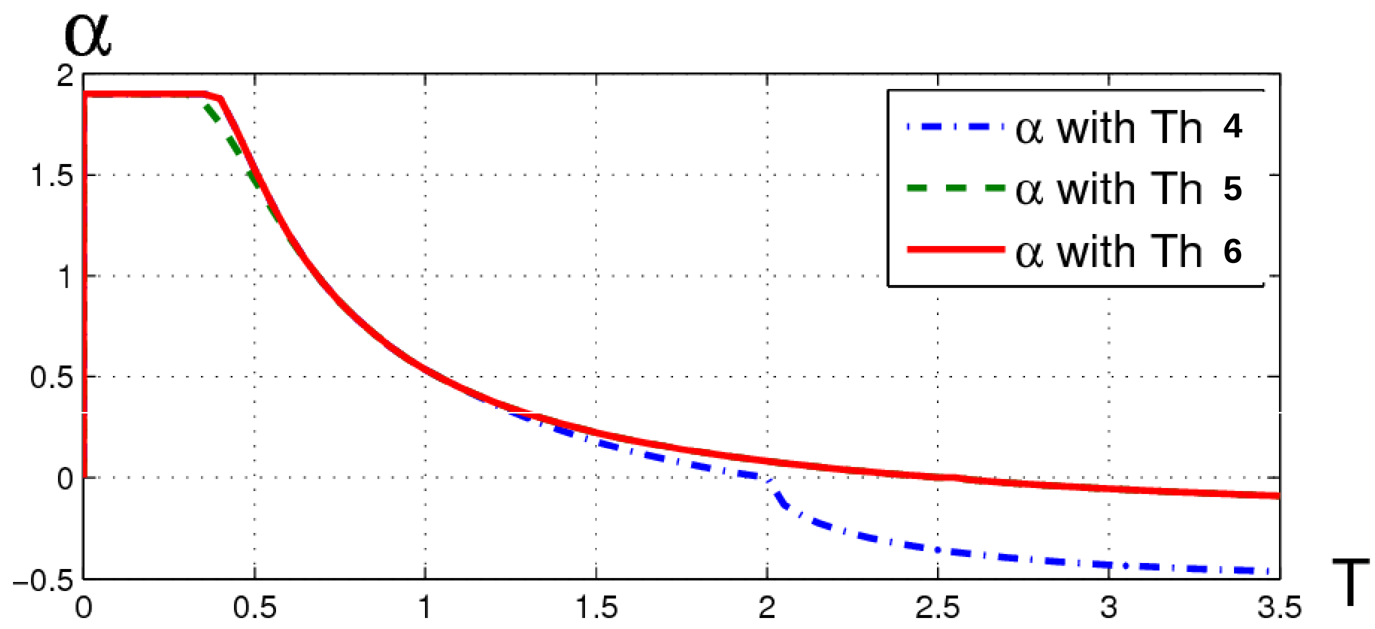
Asymptotic stability	T
Theoretical bounds	1.729
Theorem 1 (periodic or not) [Fridman et al. 2004]	0. 8696
Theorem 2 (aperiodic $(X_2 = 0)$ & periodic) [Nagahshtabrizi, 2008]	1.1137 & 1.3277
Theorem 4&5 (aperiodic & periodic)	1.719
Theorem 6 (aperiodic & periodic)	1.720
Fridman, Automatica 2010	1.695



4. Examples

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t_k)$$

Asymptotic stability	T
Theoretical bounds	~3.2
Theorem 1 (periodic or not) [Fridman et al. 2004]	0. 99
Theorem 2 (aperiodic $(X_2 = 0)$ & periodic) [Nagahshtabrizi, 2008]	1.28 & 1.61
Theorem 4 (aperiodic & periodic)	1.99
Theorem 5&6 (aperiodic & periodic)	2.51
Liu et al. 2009, Fridman 2010 (aperiodic & periodic)	2.53 & 2.03



3. Example : uncertain system

[Fridman et al, Auto, 2004], [Nagahshtabrizi, SCL. 2008]:

$$\dot{x}(t) = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix} x(t) - \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix} \begin{bmatrix} 2.6884 \\ 0.6649 \end{bmatrix}^T x(t_k)$$

where $|g_1| \leq 0.1$, and $|g_2| \leq 0.3$

Stability conditions	τ_m
Theorem 1 (periodic or not)	0.35
Theorem 2 (aperiodic $(X_2 = 0)$ & periodic)	0.44 & 0.46
Theorem 3 (aperiodic $(X_2 = 0)$ & periodic) [Liu et al, TDS'09]	0.60 & 0.70

4. Conclusion

Contribution:

- Novel and less conservative stability conditions for sampled data systems;
- Estimation of the convergence rate of sampled-data systems
- Stability of systems with uncertainties (Const. or TV)

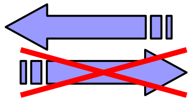
On going work

- Precise the case of ***aperiodic sampling***

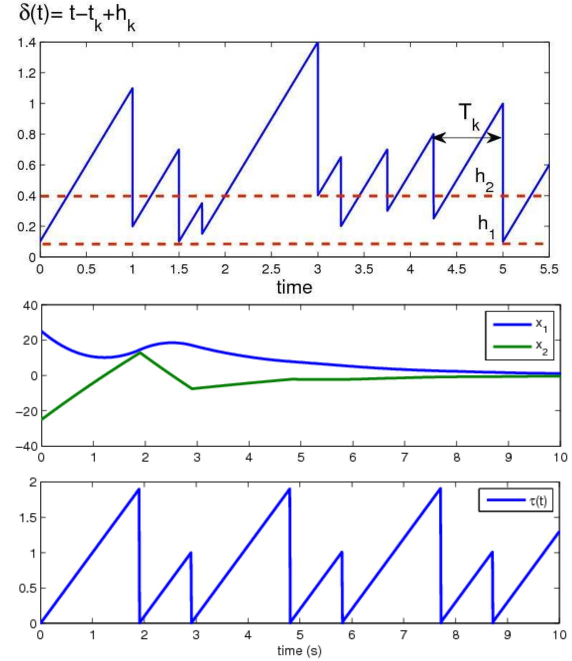
[submitted to Automatica]

$$\Delta V_k = V_{k+1} - V_k = \int_{t_k}^{t_{k+1}} \dot{V}(s)ds$$

$\Delta V_k < 0$ Discrete Lyap. Th.



$\dot{V}(s) < 0$ Cont. Lyap. Th.



•Include additional ***input delays*** (NCS)

[submitted to CDC'10 and Necsys'10]

$$\dot{x}(t) = Ax(t) + A_d x(t_k - h_k)$$

•Systems with ***several periods***
(including packet losses)

[submitted to Automatica]