# On observation of time-delay systems with unknown inputs

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# Outline

# Observability for nonlinear systems without delays

Consider the following nonlinear systems:

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

Condition: $rank \begin{pmatrix} dh \\ dL_f h \\ dL_f^2 h \\ \vdots \end{pmatrix} = n.$

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Condition: $rank \begin{pmatrix} dh \\ dL_f h \\ dL_f^2 h \\ \vdots \end{pmatrix} = n$. Ex: $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = x_3 + x_1 x_2 \\ \dot{x}_3 = x_1^2 + x_2 x_3 \\ y = x_1 \end{cases}$ Calculate the differentiation of the output:

$$\begin{aligned} dy &= dx_1 \\ d\dot{y} &= dx_1 + dx_2 \\ d\ddot{y} &= (1 + x_2)dx_1 + (1 + x_1)dx_2 + dx_3 \end{aligned}$$

# Observability for nonlinear systems with delays

$$\begin{cases} \dot{x}_1(t) = x_1(t-\tau) + x_2(t) \\ \dot{x}_2(t) = x_3(t) + x_1(t)x_2(t-2\tau) \\ \dot{x}_3(t) = x_1^2(t-\tau) + x_3(t) \\ y(t) = x_1(t) \end{cases}$$

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Quite complicated to be analyzed. Introduce delay operator $\delta$, then

$$\begin{aligned} dy(t) &= dx_1(t) \\ d\dot{y}(t) &= d(\delta x_1(t)) + dx_2(t) = \delta dx_1(t) + dx_2(t) \\ d\ddot{y}(t) &= \delta^2 x_2(t)dx_1(t) + d(\delta^2 x_1(t)) + d(\delta x_2(t)) + x_1(t)d(\delta^2 x_2(t)) + dx_3(t) \\ &= \delta^2 x_2(t)dx_1(t) + \delta^2 dx_1(t) + \delta dx_2(t) + x_1(t)\delta^2 dx_2(t) + dx_3(t) \\ &= (\delta^2 x_2 + \delta^2)dx_1 + (\delta + x_1\delta^2)dx_2 + dx_3 \end{aligned}$$

Since the coefficients are polynomials of $\delta$, we can try to establish a polynomial ring for TDS, which is not commutative.

# Time-delay systems

Consider the following nonlinear time-delay system:

$$\begin{cases} \dot{x} & = f(x(t-i\tau)) + \sum_{j=0}^{s} g^j(x(t-i\tau))u(t-j\tau) \\ y & = h(x(t-i\tau)) \\ & = [h_1(x(t-i\tau)), \dots, h_p(x(t-i\tau))]^T \\ x(t) & = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{cases} \quad (1)$$

where $x \in W \subset R^n$ denotes the state variables, $u = [u_1, \dots, u_m]^T \in R^m$ is the unknown admissible input, $y \in R^p$ is the measurable output. $p \geq m$ and $i \in S_- = \{0, 1, \dots, s\}$ is a finite set of constant time-delays.

# Non-commutative algebraic framework [4]

$\mathcal{K}$: the field of functions of a finite number of the variables from $\{x_j(t - i\tau), j \in [1, n], i \in S_-\}$.
$\mathcal{E}$: the vector space over $\mathcal{K}$: $\mathcal{E} = span_{\mathcal{K}}\{d\xi : \xi \in \mathcal{K}\}$.
$\delta$: backward time-shift operator, i.e. $\delta^i \xi(t) = \xi(t - i\tau)$ and

$$\delta^i (a(t)d\xi(t)) = \delta^i a(t)\delta^i d\xi(t)$$

$\mathcal{K}(\delta]$: the set of polynomials of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a}, a_i(t) \in \mathcal{K} \qquad (2)$$

Addition in $\mathcal{K}(\delta]$ is usual, but the multiplication is given as

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i\leq r_a, j\leq r_b} a_i(t)b_j(t - i\tau)\delta^k \qquad (3)$$

# Property

$$a(\delta] = \delta x_1 \delta, b(\delta] = x_2 + x_1 \delta^2$$

$$a(\delta] + b(\delta] = \delta x_1 \delta + x_2 + x_1 \delta^2$$

$$a(\delta] b(\delta] = \delta x_1 \delta (x_2 + x_1 \delta^2) = \delta x_1 \delta x_2 \delta + \delta x_1 \delta x_1 \delta^3$$

$$b(\delta] a(\delta] = (x_2 + x_1 \delta^2) \delta x_1 \delta = x_2 \delta x_1 \delta + x_1 \delta^3 x_1 \delta^3$$

$\mathcal{K}(\delta]$ satisfies the associative law and it is a non-commutative ring (see [4]). However, it is proved that the ring $\mathcal{K}(\delta]$ is a left Ore ring [2, 4], which enables to define the rank of a module over this ring. Let $\mathcal{M}$ denote the left module over $\mathcal{K}(\delta]$

$$\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi, \xi \in \mathcal{K}\}$$

# Time-delay systems under non-commutative rings

With the definition of $\mathcal{K}(\delta]$, (1) can be rewritten in a more compact form as follows:

$$\begin{cases} \dot{x} & = f(x,\delta) + \sum_{i=1}^{m} G_i u_i(t) \\ y & = h(x,\delta) \\ x(t) & = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{cases} \quad (4)$$

where $f(x,\delta) = f(x(t - i\tau))$ and $h(x,\delta) = h(x(t - i\tau))$ with entries belonging to $\mathcal{K}$, $G_i = \sum_{j=0}^{s} g_i^j \delta^j$ with entries belonging to $\mathcal{K}(\delta]$. It is assumed that $rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = p$, which implies that $[h_1, \ldots, h_p]^T$ are independent functions of $x$ and its backward shifts.

# Observability and Left invertibility

## Definition

System (1) is locally observable if the state $x(t)$ can be expressed as:

$$x(t) = \alpha(y(t - j\tau), \ldots, y^{(k)}(t - j\tau)) \tag{5}$$

for $j \in Z$ and $k \in Z^+$. It is locally causally observable if (5) is satisfied for $j \in Z^+$ and $k \in Z^+$, and locally non-causally observable if (5) is satisfied for $j \in Z$ and $k \in Z^+$.

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## Definition

The unknown input $u(t)$ can be estimated if it can be written as follows:

$$u(t) = \beta(y(t - j\tau), \ldots, y^{(k)}(t - j\tau)) \tag{6}$$

for $j \in Z$ and $k \in Z^+$. It can be causally estimated if (6) is satisfied for $j \in Z^+$ and $k \in Z^+$, and non-causally estimated if (6) is satisfied for $j \in Z$ and $k \in Z^+$.

# Example

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 + \delta x_1, \dot{x}_2 = \delta^2 x_2 - \delta x_3, \\ \dot{x}_3 = \delta x_4 + \delta u_1 + \delta^4 u_2, \dot{x}_4 = \delta u_2 \\ y_1 = x_1, y_2 = \delta x_4 \end{array}\right. \tag{7}$$

A straightforward calculation gives

$$\left\{\begin{array}{ll} x_1(t) & = y_1(t), x_2(t) = \dot{y}_1(t) - y_1(t-\tau) \\ x_3(t) & = \dot{y}_1(t-\tau) - y_1(t-2\tau) - \ddot{y}_1(t+\tau) + \dot{y}_1(t) \\ x_4(t) & = y_2(t+\tau) \end{array}\right.$$

and

$$\left\{\begin{array}{lll} u_1(t) & = & \ddot{y}_1(t) - \dot{y}_1(t-\tau) - \dddot{y}_1(t+2\tau) + \ddot{y}_1(t+\tau) \\ & & -y_2(t+\tau) - \dot{y}_2(t-\tau) \\ u_2(t) & = & \dot{y}(t+2\tau) \end{array}\right.$$

# Unimodular matrix and change of coordinate

## Definition

(Unimodular matrix) [3] Matrix $A \in \mathcal{K}^{n\times n}(\delta]$ is said to be unimodular over $\mathcal{K}(\delta]$ if it has a *left* inverse $A^{-1} \in \mathcal{K}^{n\times n}(\delta]$, such that $A^{-1}A = I_{n\times n}$.

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(Change of coordinate) [3] For system (1), $z = \phi(\delta, x) \in \mathcal{K}^{n\times 1}$ is a causal change of coordinate over $\mathcal{K}$ for (1) if there exists locally a function $\phi^{-1} \in \mathcal{K}^{n\times 1}$ and some constants $c_1, \cdots, c_n \in N$ such that

$$diag\{\delta^{c_i}\}x = \phi^{-1}(\delta, z).$$

The change of coordinate is bicausal over $\mathcal{K}$ if $max\{c_i\} = 0$, i.e. $x = \phi^{-1}(\delta, z)$.

# Lie derivative for TDS

Let $f(x(t-j\tau))$ and $h(x(t-j\tau))$ for $0 \leq j \leq s$ respectively be an $n$ and $p$ dimensional vector with entries $f_r \in \mathcal{K}$ for $1 \leq r \leq n$ and $h_i \in \mathcal{K}$ for $1 \leq i \leq p$. Let

$$\frac{\partial h_i}{\partial x} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathcal{K}^{1\times n}(\delta] \tag{8}$$

where for $1 \leq r \leq n$:

$$\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)}\delta^j \in \mathcal{K}(\delta]$$

then the Lie derivative for TDS can be defined as follows

$$L_f h_i = \frac{\partial h_i}{\partial x}(f) \text{ and } L_{G_i} h_i = \frac{\partial h_i}{\partial x}(G_i)$$

# Relative degree for TDS

### Definition

(Relative degree) System (4) has relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq R^n$ if, for $1 \leq i \leq p$, the following conditions are satisfied :

1. for all $x \in W$, $L_{G_j} L_f^r h_i = 0$, for all $1 \leq j \leq m$ and $0 \leq r < \nu_i - 1$;
2. there exists $x \in W$ such that $\exists j \in [1, m]$, $L_{G_j} L_f^{\nu_i - 1} h_i \neq 0$.

If for $1 \leq i \leq p$, (1) is satisfied for all $r \geq 0$, then we set $\nu_i = \infty$.

# Observability indices for TDS

Let $\mathcal{F}_k := span_{\mathcal{K}(\delta]}\left\{dh, dL_f h, \cdots, dL_f^{k-1}h\right\}$ for $1 \leq k \leq n$, satisfying $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$. then we define

$$d_1 = rank_{\mathcal{K}(\delta]}\mathcal{F}_1, \text{ and } d_k = rank_{\mathcal{K}(\delta]}\mathcal{F}_k - rank_{\mathcal{K}(\delta]}\mathcal{F}_{k-1}$$

for $2 \leq k \leq n$. Let $k_i = card\left\{d_k \geq i, 1 \leq k \leq n\right\}$ then $(k_1, \cdots, k_p)$ are the observability indices, and $\sum_{i=1}^{p} k_i = n$ since it is assumed that (4) is observable with $u = 0$. Reordering, if necessary, the output components of (4), such that

$$\begin{aligned} & rank_{\mathcal{K}(\delta]}\frac{\partial\left[h, L_f h, \cdots, L_f^{n-1}h\right]^T}{\partial x} \\ &= rank_{\mathcal{K}(\delta]}\frac{\partial\left[h_1, L_f h_1, \cdots, L_f^{k_1-1}h_1, \cdots, h_p, L_f h_p, \cdots, L_f^{k_p-1}h_p\right]^T}{\partial x} \\ &= k_1 + \cdots + k_p = n \end{aligned}$$

# Canonical form and causal observability

## Theorem 1

For $1 \leq i \leq p$, denote $k_i$ the observability indices and $\nu_i$ the relative degree index for $y_i$ of (4), and note $\rho_i = \min\{\nu_i, k_i\}$. Then there exists a change of coordinate $\phi(x, \delta) \in \mathcal{K}^{n\times 1}$, such that (4) can be transformed into the following form:

$$\dot{z}_i = A_i z_i + B_i V_i \quad (9)$$
$$\dot{\xi} = \alpha(z, \xi, \delta) + \beta(z, \xi, \delta) u \quad (10)$$
$$y_i = C_i z_i \quad (11)$$

where $A_i \in R^{\rho_i \times \rho_i}$ is in the Brounovsky form and

$$\begin{aligned} z_i &= \left(h_i, \cdots, L_f^{\rho_i - 1} h_i\right)^T \in \mathcal{K}^{\rho_i \times 1}, B_i = (0, \cdots, 0, 1)^T \in R^{\rho_i \times 1}, \\ V_i &= L_f^{\rho_i} h_i(x, \delta) + \sum_{j=1}^{m} L_{G_j} L_f^{\rho_i - 1} h_i(x, \delta) u_j \in \mathcal{K}, \alpha \in \mathcal{K}^{l\times 1} \\ \beta &\in \mathcal{K}^{l\times 1}(\delta] \text{ with } l = n - \sum_{j=1}^{p} \rho_j, C_i = (1, 0, \cdots, 0) \in R^{1\times \rho_i} \end{aligned}$$

Moreover if $k_i < \nu_i$, one has $V_i = L_f^{\rho_i} h_i = L_f^{k_i} h_i$.

For (9), note

$$H(x,\delta) = \Psi(x,\delta) + \Gamma(x,\delta)u \tag{12}$$

with

$$H(x,\delta) = \begin{pmatrix} h_1^{(\rho_1)} \\ \vdots \\ h_p^{(\rho_p)} \end{pmatrix}, \Psi(x,\delta) = \begin{pmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{pmatrix}$$

and

$$\Gamma(x,\delta) = \begin{pmatrix} L_{G_1} L_f^{\rho_1 - 1} h_1 & \cdots & L_{G_m} L_f^{\rho_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p - 1} h_p & \cdots & L_{G_m} L_f^{\rho_p - 1} h_p \end{pmatrix}$$

where $H(x,\delta) \in \mathcal{K}^{p\times 1}$, $\Psi(x,\delta) \in \mathcal{K}^{p\times 1}$ and $\Gamma(x,\delta) \in \mathcal{K}^{p\times m}(\delta]$.
And for (4), let denote $\Phi$ the observable space from its outputs:

$$\Phi = \{dh_1, \cdots, dL_f^{\rho_1 - 1} h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p - 1} h_p\} \tag{13}$$

# Main theorem

## Theorem 2

For system (4) with outputs $(y_1, \ldots, y_p)$ and corresponding $(\rho_1, \ldots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where $k_i$ and $\nu_i$ are respectively the associated observability indices and the relative degree, if

$$rank_{\mathcal{K}(\delta]} \Phi = n$$

where $\Phi$ defined in (13), then there exists a change of coordinate $\phi(x, \delta)$ such that (4) can be transformed into (9-11) with $dim \xi = 0$.
Moreover, if the change of coordinate is locally bicausal over $\mathcal{K}$, then the state $x(t)$ of (4) is locally causally observable.
For the matrix $\Gamma \in \mathcal{K}^{p \times m}(\delta]$ where $m \leq p$, if $rank_{\mathcal{K}(\delta]} \Gamma = m$, then there exists a matrix $Q \in \mathcal{K}^{p \times p}(\delta]$ such that $Q\Gamma = \begin{bmatrix} \bar{\Gamma} \\ \mathbf{0} \end{bmatrix}$ where $\bar{\Gamma} \in \mathcal{K}^{m \times m}(\delta]$ has full row rank $m$. Moreover, if $\bar{\Gamma} \in \mathcal{K}^{m \times m}(\delta]$ is also unimodular over $\mathcal{K}(\delta]$, then the unknown input $u(t)$ of (4) can be causally estimated.

# Example

$$\begin{cases} \dot{x}_1 = -\delta x_1 + x_2, \dot{x}_2 = -\delta x_3 + u_1 \\ \dot{x}_3 = \delta x_1 + \delta u_1 + u_2, \dot{x}_4 = -x_4 + 2\delta x_4/3 \\ y_1 = x_1, y_2 = x_3, y_3 = x_4 \end{cases} \tag{14}$$

$\Rightarrow \nu_1 = k_1 = 2,\ \nu_2 = k_2 = 1,\ \nu_3 = \infty$ and $k_3 = 1 \Rightarrow \rho_1 = 2,\ \rho_2 = 1$ and $\rho_3 = 1 \Rightarrow$

$$\Phi = \{dh_1, dL_f h_1, dh_2, dh_3\} = \{dx_1, -\delta dx_1 + dx_2, dx_3, dx_4\}$$

$\Rightarrow rank_{\mathcal{K}(\delta]}\Phi = 4 \Rightarrow$

$$z = \phi(x, \delta) = (x_1, x_2 - \delta x_1, x_3, x_4)^T$$

# Example

Since

$$x = \phi^{-1} = (z_1, \delta z_1 + z_2, z_3, z_4, )^T$$

$\Rightarrow$ the change of coordinate is bicausal over $\mathcal{K}$, thus the state of (14) is locally causally observable:

$$\begin{cases} x_1(t) & = y_1(t), x_2(t) = y_1(t-\tau) + \dot{y}_1(t) \\ x_3(t) & = y_2(t), x_4(t) = y_3(t) \end{cases}$$

Moreover, since $\Gamma = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Rightarrow \Gamma^{-1} = \begin{pmatrix} 1 & 0 \\ -\delta & 1 \end{pmatrix}$ s.t. $\Gamma^{-1}\Gamma = I_{2\times 2}$

$\Rightarrow$ $\Gamma$ is unimodular over $\mathcal{K}(\delta]$$\Rightarrow$the unknown inputs are causally observable as well:

$$\begin{cases} u_1(t) & = & \dot{y}_1(t-\tau) + \ddot{y}_1(t) + y_2(t-\tau) \\ u_2(t) & = & \dot{y}_2(t) - y_1(t-\tau) - \dot{y}_1(t-2\tau) - \ddot{y}_1(t-\tau) - y_2(t-2\tau) \end{cases}$$

# Remark

- The condition of $rank_{\mathcal{K}(\delta]}\Phi = n$ is sometimes hard to be satisfied.
- When $rank_{\mathcal{K}(\delta]}\Phi < n$, is it still possible to estimate the state and the unknown inputs?

In [1], a constructive algorithm to solve this problem for nonlinear systems without delays has been proposed, which in fact can be generalized to treat the same problem for nonlinear time-delay systems.

# Illustrative example

Ex:

$$\begin{cases} \dot{x}_1 = -\delta x_1 + \delta x_4 u_1, \dot{x}_2 = -\delta x_3 + x_4 \\ \dot{x}_3 = x_2 - \delta x_4 u_1, \dot{x}_4 = u_2 \\ y_1 = x_1, y_2 = \delta x_1 + x_3 \end{cases} \tag{15}$$

$\Rightarrow \nu_1 = k_1 = 1,\ \nu_2 = 1,\ k_2 = 3 \Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3\}$
$\Rightarrow rank_{\mathcal{K}(\delta]}\Phi = 2 < n \Rightarrow$ Theorem 2 cannot be applied.
Precisely,

$$\dot{y}_1 = -\delta x_1 + \delta x_4 u_1$$

and $\dot{y}_2 = -\delta^2 x_1 + \delta^2 x_4 \delta u_1 + x_2 - \delta x_4 u_1 \Rightarrow$ derivative impossible.

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Precisely,

$$\dot{y}_1 = -\delta x_1 + \delta x_4 u_1$$

and $\dot{y}_2 = -\delta^2 x_1 + \delta^2 x_4 \delta u_1 + x_2 - \delta x_4 u_1 \Rightarrow$derivative impossible. However,

$\dot{y}_2 - (\delta - 1)\dot{y}_1 \Rightarrow \ \dot{y}_2 - (\delta - 1)\dot{y}_1 = -\delta x_1 + x_2 \ \Rightarrow x_2 = \dot{y}_2 - (\delta - 1)\dot{y}_1 + \delta y_1$

Note $y_3 = x_2 \Rightarrow \nu_3 = k_3 = 2 \Rightarrow \rho_3 = 2 \Rightarrow$
$\Phi = \{dx_1, \delta dx_1 + dx_3, dx_2, -\delta dx_3 + dx_4\} \Rightarrow rank_{\mathcal{K}(\delta]}\Phi = 4$

# Notation and Definition

For the case where $rank_{\mathcal{K}(\delta]}\Phi = j < n$, select $j$ linearly independent vector over $R[\delta]$ from $\Phi$, where $R[\delta]$ means the set of polynomials of $\delta$ with coefficients belonging to $R$, noted as

$$\Phi \quad = \{dz_1, \cdots, dz_j\}$$

Note

$$\mathcal{L} = span_{R[\delta]}\left\{z_1, \cdots, z_j\right\}$$

and let $\mathcal{L}(\delta]$ be the set of polynomials of $\delta$ with coefficients over $\mathcal{L}$, define the module spanned by element of $\Phi$ over $\mathcal{L}(\delta]$ as follows

$$\Omega = span_{\mathcal{L}(\delta]}\left\{\xi, \xi \in \Phi\right\} \tag{16}$$

Define $\mathcal{G} = span_{R[\delta]}\{G_1, \ldots, G_m\}$ and its *left* annihilator

$$\mathcal{G}^{\perp} = span_{R[\delta]}\{\omega \in \Omega \mid \omega g = 0, \forall g \in \mathcal{G}\}$$

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# Theorem for general case

## Theorem 3

For (4) with outputs $y = (y_1, \cdots, y_p)^T$ and corresponding $(\rho_1, \ldots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where $k_i$ and $\nu_i$ are respectively the associated observability indices and the relative degree which yields (12) with $rank_{\mathcal{K}(\delta]}\Phi < n$ where $\Phi$ is defined in (13), there exists $l$ new independent outputs over $\mathcal{K}$ which are functions of $y$ and its time derivatives and backwards time shifts, if and only if $rank_{\mathcal{K}}\mathcal{H} = l$ where

$$\mathcal{H} = span_{R[\delta]}\{\omega \in \mathcal{G}^{\perp} \cap \Omega \mid \omega f \notin \mathcal{L}\} \tag{17}$$

with $\Omega$ defined in (16).

Moreover, the new outputs, noted $\bar{y}_i$ for $1 \leq i \leq l$, are given as follows:

$$\bar{y}_i = \omega_i f \mod \mathcal{L}$$

where $\omega_i \in \mathcal{H}$.

# Remarks

- Roughly speaking, for

$$H(x, \delta) = \Psi(x, \delta) + \Gamma(x, \delta)u$$

  if there exists a $1 \times p$ vector $Q$ with entries $q_i \in \mathcal{L}(\delta]$, such that $Q\Gamma = 0$ and $Q\Psi \notin \mathcal{L}$, then we denote

$$y_{p+1} = Q\Psi \mod \mathcal{L}$$

  a new output since it is not affected by the unknown input $u$, and it does not belong to the current measurable vector $\mathcal{L}$.
- Theorem 3 gives a constructive way to treat the case where $rank_{\mathcal{K}(\delta]}\Phi < n$.
- A 'Check-Extend' procedure is iterated until one obtains $rank_{\mathcal{K}(\delta]}\Phi = n$.

# Routine to deduce the new outputs

**Input**: DTS with $x \in R^n$, $y \in R^p$, $u \in R^m$
**Output**: $\Phi$ *or failed*
**Initialization**: Compute $\nu_i$, $k_i$, $\rho_i$, $\Phi$, $rank_{\mathcal{K}(\delta]}\Phi = j$
**Loop:**

**While** $j < n$

$\Phi = \left\{dz_1, \cdots, dz_j\right\}$
$\mathcal{L} = span_{R[\delta]} \left\{z_1, \cdots, z_j\right\}$
$\Omega = span_{\mathcal{L}(\delta]} \{\xi, \xi \in \Phi\}$
$\mathcal{H} = span_{R[\delta]}\{\omega \in \mathcal{G}^{\perp} \cap \Omega \mid \omega f \notin \mathcal{L}\}$
$rank_{\mathcal{K}(\delta]}\mathcal{H} = l$
**If** $l > 0$

$\exists l$ 1-forms, s.t. $\mathcal{H} = span_{R[\delta]} \{\omega_1, \cdots \omega_l\}$
$y = y \cup \{\omega_i f \mod \mathcal{L}, 1 \leq i \leq l\}$
Reorder $y$
For each $y_i \in y$, calculate $\nu_i$, $k_i$, $\rho_i$
$\phi = \{\cdots, dh_i, \cdots, dL_f^{\rho_i} h_i, \cdots\}$
$rank_{\mathcal{K}(\delta]}\Phi = j$

**Else**

**Return**(failed)

**End**

**Return**($\Phi$)

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# Example

$$\begin{cases} \dot{x}_1 = -\delta x_1 + \delta x_4 u_1, \dot{x}_2 = -\delta x_3 + x_4 \\ \dot{x}_3 = x_2 - \delta x_4 u_1, \dot{x}_4 = u_2 \\ y_1 = x_1, y_2 = \delta x_1 + x_3 \end{cases} \quad (18)$$

$\Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3\} \Rightarrow rank_{\mathcal{K}(\delta]}\Phi = 2 < n.$
$\mathcal{G} = span_{R[\delta]}\{(\delta x_4, 0, -\delta x_4, 0)^T, (0, 0, 0, 1)^T\} \Rightarrow \mathcal{G}^\perp =$
$span_{R[\delta]}\{dx_1 + dx_3, dx_2\}$
$rank_{\mathcal{K}(\delta]}\Phi = 2 \Rightarrow \mathcal{L} = span_{R[\delta]}\{x_1, \delta x_1 + x_3\} \Rightarrow \Omega = span_{\mathcal{L}(\delta]}\{dx_1, dx_3\}$

$$\begin{aligned} \Omega \cap \mathcal{G}^\perp &= span_{\mathcal{L}(\delta]}\{dx_1, dx_3\} \cap span_{R[\delta]}\{dx_1 + dx_3, dx_2\} \\ &= span_{\mathcal{L}(\delta]}\{dx_1 + dx_3\} \end{aligned}$$

$\Rightarrow \forall \omega \in \Omega \cap \mathcal{G}^\perp, \omega f \notin \mathcal{L}$ since $\omega f = -\delta x_1 + x_2 \Rightarrow$ new output $h_3$:

$$y_3 = h_3 = \omega f \mod \mathcal{L} = x_2 = \delta y_1 + (1 - \delta)\dot{y}_1 + \dot{y}_2 \quad (19)$$

# Example

$\Rightarrow \rho_1 = \rho_2 = 1$ and $\rho_3 = 2 \Rightarrow \Phi = \{dx_1, \delta dx_1 + dx_3, dx_2, -\delta dx_3 + dx_4\}$
$\Rightarrow rank_{\mathcal{K}(\delta]}\Phi = 4 = n$, thus we find the following change of coordinate

$$z = \phi(x, \delta) = (x_1, \delta x_1 + x_3, x_2, -\delta x_3 + x_4)^T$$

it is easy to check that it is bicausal over $\mathcal{K}(\delta]$, since

$$x = \phi^{-1} = \left(z_1, z_3, z_2 - \delta z_1, z_4 + \delta z_2 - \delta^2 z_1\right)$$

and one gets

$$\begin{cases} x_1 = y_1, x_2 = y_3, x_3 = y_2 - \delta y_1, \\ x_4 = -\delta^2 y_1 + \delta y_2 + \dot{y}_3 \end{cases}$$

where the new output $y_3$ is defined in (19).

$$\begin{cases} u_1 = \frac{\dot{y}_1}{-\delta^3 y_1 + \delta^2 y_2 + \delta \dot{y}_3} \\ u_2 = -\delta \dot{y}_1 + \ddot{y}_3 \end{cases}$$

# Remark

- It is the locally bicausal change of coordinate which makes the state of system locally causally observable.
- It is the unimodular characteristic of $\Gamma$ over $\mathcal{K}(\delta]$ which guarantees the causal reconstruction of unknown inputs.

The following is devoted to dealing with the non-causal case.

# Non-causal observability

$\nabla$: the forward time-shift operator, such that for $i, j \in N$,

$$\nabla f(t) = f(t+\tau), \nabla^i \delta^j f(t) = \delta^j \nabla^i f(t) = f\left(t-(j-i)\tau\right)$$

$\bar{\mathcal{K}}$: the field of functions of a finite number of variables from $\{x_j(t-i\tau), j \in [1,n], i \in S\}$ where $S = \{-s, \cdots, 0, \cdots, s\}$
$\bar{\mathcal{K}}(\delta, \nabla]$: the set of polynomials of the following form:

$$a(\delta, \nabla] = \bar{a}_{r_{\bar{a}}} \nabla^{r_{\bar{a}}} + \cdots + \bar{a}_1 \nabla + a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{20}$$

where $a_i(t)$ and $\bar{a}_i(t)$ belonging to $\bar{\mathcal{K}}$.
Usual addition + the following multiplication:

$$a(\delta,\nabla]b(\delta,\nabla] = \sum_{i=0}^{r_a}\sum_{j=0}^{r_b} a_i\delta^i b_j \delta^{i+j} + \sum_{i=0}^{r_a}\sum_{j=1}^{r_{\bar{b}}} a_i\delta^i \bar{b}_j \delta^i \nabla^j + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=0}^{r_b} \bar{a}_i \nabla^i b_j \nabla^i \delta^j + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=1}^{r_{\bar{b}}} \bar{a}_i \nabla^i \bar{b}_j \nabla^{i+j}$$

It is clear that $\mathcal{K} \subseteq \bar{\mathcal{K}}$ and $\mathcal{K}(\delta] \subseteq \bar{\mathcal{K}}(\delta, \nabla]$.

# Theorem

### Theorem 4

For system (4) with outputs $(y_1, \ldots, y_p)$ and corresponding $(\rho_1, \ldots, \rho_p)$, if $rank_{\mathcal{K}(\delta]}\Phi = n$, where $\Phi$ defined in (13), then there exists a change of coordinate $z = \phi(x, \delta)$ such that (4) can be transformed into (9-11) with $dim\xi = 0$.
Moreover, if the change of coordinate $z = \phi(x, \delta)$ is locally bicausal over $\bar{\mathcal{K}}$, then the state $x(t)$ of (4) is at least non-causally observable.
For the deduced matrix $\Gamma$ with $rank_{\mathcal{K}(\delta]}\Gamma = m$, one can obtain a matrix $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ which has full row rank $m$. If $\bar{\Gamma}$ is unimodular over $\bar{\mathcal{K}}(\delta, \nabla]$, then the unknown input $u(t)$ of (4) can be at least non-causally estimated as well. ■

# Example

$$\begin{cases} \dot{x}_1 = \delta x_1 + x_2\delta u_1, \dot{x}_2 = -x_1 + u_1 + x_3\delta u_2 \\ \dot{x}_3 = x_4 - x_1\delta x_2\delta^2 u_1, \dot{x}_4 = \delta x_1 + \delta^3 x_2 \\ y_1 = x_1, y_2 = \delta x_3 \end{cases} \tag{21}$$

$\Rightarrow \nu_1 = k_1 = 1,\ \nu_2 = 1,\ k_2 = 3 \Rightarrow \rho_1 = \rho_2 = 1 \Rightarrow \Phi = \{dx_1, \delta dx_3\} \Rightarrow$ $rank_{\mathcal{K}(\delta]}\Phi = 2 < n.$

$$\mathcal{G} = span_{R[\delta]}\{G_1, G_2\} \Rightarrow \mathcal{G}^{\perp} = span_{R[\delta]}\{x_1\delta dx_1 + dx_3, dx_4\}$$

$rank_{\mathcal{K}(\delta]}\Phi = 2 \Rightarrow \mathcal{L} = span_{R[\delta]}\{x_1, \delta x_3\} \Rightarrow \Omega = span_{\mathcal{L}(\delta]}\{dx_1, \delta dx_3\} \Rightarrow$

$$\begin{aligned} \Omega \cap \mathcal{G}^{\perp} &= span_{\mathcal{L}(\delta]}\{dx_1, \delta dx_3\} \cap span_{R[\delta]}\{x_1\delta dx_1 + dx_3, dx_4\} \\ &= span_{\mathcal{L}(\delta]}\{\delta x_1\delta^2 dx_1 + \delta dx_3\} \end{aligned}$$

Since $\omega f = \delta x_1\delta^3 x_1 + \delta x_4 \notin \mathcal{L} \Rightarrow$

$$y_3 = h_3 = \omega f \mod \mathcal{L} = \delta x_4 = \delta y_1\delta^2\dot{y}_1 + \dot{y}_2 - \delta y_1\delta^3 y_1 \tag{22}$$

# Example

$\Rightarrow \rho_1 = \rho_2 = 1, \nu_3 = k_3 = 2 \Rightarrow \rho_3 = 2 \Rightarrow \Phi = \left\{dx_1, \delta dx_3, \delta dx_4, \delta^3 dx_2\right\}$
$\Rightarrow rank_{\mathcal{K}(\delta)}\Phi = 4 = n \Rightarrow \mathcal{L} = span_{R[\delta]}\{x_1, \delta x_3, \delta x_4, \delta^3 x_2\} \Rightarrow$ the following change of coordinate

$$z = \phi(x, \delta) = \left(x_1, \delta x_3, \delta x_4, \delta x_1 + \delta^3 x_2\right)^T$$

which is not bicausal over $\mathcal{K}$, but bicausal over $\bar{\mathcal{K}}$, since one has

$$x = \phi^{-1} = \left(z_1, -\nabla^2 z_1 + \nabla^3 z_4, \nabla z_2, \nabla z_3\right)^T$$

which gives

$$\begin{cases} x_1 = y_1, x_2 = -\nabla^2 y_1 + \nabla^3 \dot{y}_3 \\ x_3 = \nabla y_2, x_4 = \nabla y_3 \end{cases}$$

where $y_3$ is given in (22). Thus $x$ of (21) is observable, but non causally observable. The calculation for $u$ is omitted (see [5] for more details).

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