Reduction and decomposition of linear differential time-delay systems

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Smith/Jacobson normal form

- Let D be a principal left/right ideal domain.
- Example: $\mathbb{R}[s]$, $K\langle\partial\rangle$, K differential field (e.g., $\mathbb{Q}(t)$, $\mathbb{R}\{t\}[t^{-1}]$).
- $\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\; U\,V = V\,U = I_p\}$.
- Theorem: $\forall\, R \in D^{q\times p}$, $\exists\, V \in \mathrm{GL}_q(D)$, $\exists\, U \in \mathrm{GL}_p(D)$:

$$\overline{R} := V\,R\,U = \left(\begin{array}{ccccccc} \alpha_1 & 0 & \ldots & \ldots & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \ldots & 0 & \alpha_r & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & \ldots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & \ldots & 0 \end{array}\right),$$

where $\alpha_1 \,||\, \alpha_2 \,||\, \ldots \,||\, \alpha_r \neq 0$ (see, e.g., Kailath 80, Cohn 03).

Example: Smith normal form

- Let us consider 2 pendulum of the same length mounted on a car:

$$\begin{cases} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{cases} \qquad \alpha = g/l.$$

- Let us consider the principal ideal domain $D = \mathbb{Q}(\alpha)[\partial]$.

$$P = \sum_{i=0}^{n} a_i\, \partial^i \in D, \quad a_i \in \mathbb{Q}(\alpha), \quad \partial\, y(t) = \dot{y}(t).$$

$$\begin{pmatrix} -\alpha & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial^2 + \alpha & 0 & -\alpha \\ 0 & \partial^2 + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & \partial^2 + \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^2 + \alpha & 0 \end{pmatrix}.$$

Example: Jacobson normal form

- Let us consider the time-varying linear system:

$$\begin{cases} t\,\dot{y}_1(t) - y_1(t) - t^2\,\dot{y}_2(t) + u_1(t) = 0, \\ \dot{y}_1(t) + t\,\dot{y}_2(t) - y_2(t) + u_2(t) = 0. \end{cases}$$

- Let us consider the principal left ideal domain $D = \mathbb{Q}(t)\langle\partial\rangle$.

$$\begin{pmatrix} t\,\partial - 1 & -t^2\,\partial & 1 & 0 \\ \partial & t\,\partial - 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t\,\partial + 1 & t^2\,\partial \\ 0 & 1 & -\partial & -t\,\partial + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

- Implementation in the Maple package JACOBSON (Culianez, Q.).

Rings of ordinary differential operators

- Ordinary differential equations: (Boole 1859)

$$\sum_{i=1}^{n} a_i(t)\, y^{(i)}(t) = 0 \;\Leftrightarrow\; \left(\sum_{i=1}^{n} a_i(t)\, \partial^i\right) y(t) = 0,$$

$$\partial\colon y \longmapsto \frac{dy}{dt}, \quad \partial^i = \partial \circ \cdots \circ \partial, \quad a_i = a_i(\cdot), \quad a_i\colon y \longmapsto a_i\, y,$$

$\Rightarrow$ noncommutative polynomial rings of OD operators:

Product: $$\left(\sum_{j=0}^{m} b_j(t)\, \partial^j\right) \left(\sum_{i=0}^{n} a_i(t)\, \partial^i\right) = \sum_{k=0}^{m+n} c_k(t)\, \partial^k$$

$$(\partial \circ a)(y) = \partial\,(a(y)) = \partial(a\, y) = \frac{d}{dt}(a\, y) = a\, \frac{dy}{dt} + \frac{da}{dt}\, y$$
$$= \left(a \circ \partial + \frac{da}{dt}\right)(y) \quad \Rightarrow \quad \partial\, a = a\, \partial + \frac{da}{dt}.$$

Rings of functional operators

- Ordinary differential operator: (Boole 1859)

$$\partial\colon y \longmapsto \frac{dy}{dt}, \quad a = a(\cdot), \quad a\colon y \longmapsto a\,y, \quad (\partial\, a)(y) = \left(a\,\partial + \frac{da}{dt}\right)(y).$$

- Shift operator: (Boole 1872)

$$\partial\colon y_n \longmapsto \sigma(y_n) = y_{n+1}, \quad a = (a_n)_{n\in\mathbb{N}}, \quad a\colon y_n \longmapsto a_n\,y_n,$$

$$(\partial\, a)(y_n) = \partial(a(y_n)) = \partial(a_n\,y_n) = \sigma\,(a_n\,y_n) = a_{n+1}\,y_{n+1} = (\sigma(a)\,\partial)(y_n).$$

- Time-delay operator:

$$\partial\colon y \longmapsto \delta(y) = y(\cdot - h), \quad h \in \mathbb{R}_+,$$

$$(\partial\, a)(y) = \partial(a(y)) = \partial(a\,y) = \delta\,(a\,y) = a(\cdot - h)\,y(\cdot - h) = (\delta(a)\,\partial)(y).$$

- Euler, Frobenius, difference, q-difference, q-shift, q-dilation...

Skew polynomial rings

- Definition (Ore 33): A skew polynomial ring $A[\partial;\alpha,\beta]$ is a noncommutative polynomial ring in ∂ with coefficients in a domain A satisfying

$$\forall\, a \in A, \quad \partial\, a = \alpha(a)\,\partial + \beta(a),$$

where $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are such that

$$\left\{\begin{array}{l} \alpha(1) = 1, \\ \alpha(a+b) = \alpha(a) + \alpha(b), \\ \alpha(a\,b) = \alpha(a)\,\alpha(b), \end{array}\right. \qquad \left\{\begin{array}{l} \beta(a+b) = \beta(a) + \beta(b), \\ \beta(a\,b) = \alpha(a)\,\beta(b) + \beta(a)\,b, \end{array}\right.$$

i.e., α is an endomorphism of A and β is a α-derivation of A.

- $P \in A[\partial;\alpha,\beta]$ has a unique form $P = \sum_{i=0}^{n} a_i\,\partial^i, \; a_i \in A$.
  - Ring of ordinary differential operators: $A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.
  - Ring of shift operators: $A[\partial;\delta,0]$, $A[\partial;\sigma,0]$.

Ore extensions and Ore algebras

- We can iterate skew polynomial rings to get Ore extensions:

$$A[\partial_1;\alpha_1,\beta_1]\ldots[\partial_n;\alpha_n,\beta_n] := (((A[\partial_1;\alpha_1,\beta_1])[\partial_2;\alpha_2,\beta_2])\ldots)[\partial_n;\alpha_n,\beta_n].$$

- Definition (Chyzak-Salvy 96): $A[\partial_1;\alpha_1,\beta_1]\ldots[\partial_n;\alpha_n,\beta_n]$ is called an Ore algebra if the ∂_i's commute, i.e.:

$$\forall\, i,j=1,\ldots,n, \quad \partial_i\,\partial_j=\partial_j\,\partial_i.$$

  - Ring of differential operators: $A\left[\partial_1;\mathrm{id},\frac{\partial}{\partial x_1}\right]\ldots\left[\partial_n;\mathrm{id},\frac{\partial}{\partial x_n}\right]$.
  - Ring of differential delay operators: $A\left[\partial_1;\mathrm{id},\frac{d}{dt}\right][\partial_2;\delta,0]$:

$$(\partial_2\,\partial_1)(y(\cdot))=\partial_2(\dot{y}(\cdot))=\dot{y}(\cdot-h)=\partial_1(y(\cdot-h))=(\partial_1\,\partial_2)(y(\cdot)).$$

  - Ring of multi-shift operators: $A[\partial_1;\sigma_1,0]\ldots[\partial_n;\sigma_n,0]$.

Matrix of functional operators

- Wind tunnel model (Manitius 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{array}\right. \qquad (\star)$$

- Let $D = \mathbb{Q}(a, k, \omega)[\partial, \delta]$ be the commutative polynomial ring:

$$\partial\,x(t) = \dot{x}(t), \quad \delta\,x(t) = x(t-h).$$

$$(\partial\,\delta)(x(t)) = \partial(x(t-h)) = \dot{x}(t-h) = \delta(\dot{x}(t)) = (\delta\,\partial)(x(t)).$$

- The linear differential time-delay system $(\star)$ can be rewritten as:

$$\left(\begin{array}{cccc} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{array}\right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{array}\right) = 0.$$

Noncommutative Gröbner bases

- Theorem (Kredel 93): Let k be a computable field ($k = \mathbb{Q}$, $\mathbb{F}_p$), $A = k[x_1, \ldots, x_n]$ and $D = A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_m; \alpha_m, \beta_m]$ such that

$$\alpha_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$.

Then, a noncommutative version of Buchberger's algorithm terminates for any term order and its result is a Gröbner basis.

- Possible extensions to the case of $A = k(x_1, \ldots, x_n)$.
- Implementation in the Maple package Ore_algebra (Chyzak).
- Gröbner bases can be used to effectively compute over D.
- For more details, see Chyzak-Q.-Robertz, Effective algorithms for parametrizing linear control systems over Ore algebras, AAECC 05.

Factorization, reduction and decomposition problems

- $\mathrm{GL}_p(D) = \{U \in D^{p \times p} \mid \exists\, V \in D^{p \times p} : \; U\,V = V\,U = I_p\}$.

1. $\exists\, R_1 \in D^{r \times p},\; R_2 \in D^{q \times r} : R = R_2\, R_1\,$?
2. $\exists\, W \in \mathrm{GL}_p(D),\; V \in \mathrm{GL}_q(D) : V\,R\,W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$?
3. $\exists\, W \in \mathrm{GL}_p(D),\; V \in \mathrm{GL}_q(D) : V\,R\,W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$?
4. $\exists\, W \in \mathrm{GL}_p(D),\; V \in \mathrm{GL}_q(D) : V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & S_{22} \end{pmatrix}$?

- Problem 4 is called Serre's reduction (Serre 60-61).

Motivations

- Let $\mathcal{F}$ be a left D-module:

$$\forall\, d_1,\, d_2 \in D, \quad \forall\, f_1,\, f_2 \in \mathcal{F}: \quad d_1\, f_1 + d_2\, f_2 \in \mathcal{F}.$$

- If $R \in D^{q \times p}$, then we can define the linear system/behaviour:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F} \mid R\,\eta = 0\}.$$

1. $R = R_2\, R_1 \quad \Rightarrow \quad \ker_{\mathcal{F}}(R_1.) \subseteq \ker_{\mathcal{F}}(R.)$.
2. $V\,R\,W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \quad \Rightarrow \quad$ cascade integration:

$$S_{22}\,\zeta_2 = 0 \quad \Rightarrow \quad S_{11}\,\zeta_1 = -S_{22}\,\zeta_2.$$

3. $S_{12} = 0 \quad \Rightarrow \quad \ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(S_{11}.) \oplus \ker_{\mathcal{F}}(S_{22}.)$.
4. $S_{12} = 0$ & $S_{11} = I_r \quad \Rightarrow \quad \ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(S_{22}.)$.

Algebraic analysis (≥ 60, Malgrange, Sato, Kashiwara...)

- Let D be an Ore algebra over a noetherian domain A, $R \in D^{q\times p}$.
- Let us consider the left D-homomorphism (D-linear map):

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1 \ \dots \ \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$M := \operatorname{coker}_D(.R) = D^{1\times p}/\operatorname{im}_D(.R) = D^{1\times p}/(D^{1\times q}\, R).$$

- M is the left D-module formed by the equivalence classes $\pi(\mu)$ of $\mu \in D^{1\times p}$ for the equivalence relation $\sim$ on $D^{1\times p}$:

$$\mu_1 \sim \mu_2 \ \Leftrightarrow \ \exists\, \lambda \in D^{1\times q} : \ \mu_1 = \mu_2 + \lambda\, R.$$

  - Number theory: $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$, $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2+5)$.
  - Algebraic geometry: $\mathbb{C}[x,y]/(x^2+y^2-1, x-y)$.

Duality: modules – linear systems

- Let $\{f_k\}_{k=1,\ldots,p}$ the standard basis of $D^{1\times p}$ ($f_k = (0\ldots 1\ldots 0)$).
- Let $\pi : D^{1\times p} \longrightarrow M$ be the D-homomorphism sending μ to $\pi(\mu)$.
- $M = D^{1\times p}/(D^{1\times q}\, R)$ is generated by $\{y_k = \pi(f_k)\}_{k=1,\ldots,p}$ and:

$$l = 1,\ldots,q, \quad \sum_{k=1}^{p} R_{lk}\, y_k = 0 \;\Leftrightarrow\; R\, y = 0, \quad y = (y_1 \;\ldots\; y_p)^T.$$

- Let $\mathcal{F}$ be a left D-module and the linear functional system:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \quad \text{(behaviour)}.$$

- Let $\hom_D(M,\mathcal{F})$ be the abelian group of D-homomorphisms:

$$\hom_D(M,\mathcal{F}) := \{f : M \to \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- Remark & Theorem (Malgrange 62): $\hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R.)$.

Example: Wind tunnel model

- Let $D = \mathbb{Q}(a, k, \omega)[\partial, \delta]$ be the commutative polynomial ring,

$$R = \begin{pmatrix} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$.

- This module was first introduced by Mounier 95.

- If $\mathcal{F}$ is a D-module (e.g., $\mathcal{F} = C^{\infty}(\mathbb{R})$), then:

$$\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta = (x_1 \quad x_2 \quad u)^T \in \mathcal{F}^3 \mid R\,\eta = 0\}, \text{i.e.:}$$

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases}$$

Homomorphisms and transformations

- $R' \in D^{q' \times p'}$, $\quad M' = D^{1\times p'}/(D^{1\times q'}\, R')$, $\quad \pi' : D^{1\times p'} \longrightarrow M'$.

- Lemma: $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$, where $P \in D^{p\times p'}$ and $Q \in D^{q \times q'}$ are such that:

$$R\,P = Q\,R'.$$

- If $f \in \hom_D(M, M')$, then we have $(R\,P\,\eta' = Q\,R'\,\eta' = 0)$:

$$\begin{array}{rcl} f^\star : \ker_{\mathcal{F}}(R'.) & \longrightarrow & \ker_{\mathcal{F}}(R.), \\ \eta' & \longmapsto & \eta = P\,\eta'. \end{array}$$

- $\hom_D(M, M')$ can be totally (resp., partially) computed if D is a commutative (resp., noncommutative) polynomial ring.

- $\mathrm{end}_D(M) := \hom_D(M, M)$ is the internal symmetries of M

  $\Rightarrow$ Galois-like transformations of $\ker_{\mathcal{F}}(R.)$.

- Maple package OreMorphisms (Cluzeau-Q.).

Contravariant left exact functor $\hom_{\mathcal{F}}(\,\cdot\,, \mathcal{F})$

- $\hom_{\mathcal{F}}(\,\cdot\,, \mathcal{F}) : \; {}_D\mathrm{Mod}^{\mathrm{fp}} \longrightarrow \mathrm{Ab}$ (or Vec_k).

Modules | Linear systems/behaviours

$$\begin{array}{ccc} D^{1\times p}/(D^{1\times q}\,R) = M & \xrightarrow{\hom_D(\,\cdot\,,\mathcal{F})} & \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R.) \\ \downarrow f & & \uparrow f^{\star} \\ D^{1\times p'}/(D^{1\times q'}\,R') = M' & \xrightarrow{\hom_D(\,\cdot\,,\mathcal{F})} & \hom_D(M',\mathcal{F}) \cong \ker_{\mathcal{F}}(R'.) \end{array}$$

Algebra | "Geometry"

Internal symmetries | Galois-like transformations

$$\begin{array}{ccc} D^{1\times p}/(D^{1\times q}\,R) = M & \xrightarrow{\hom_D(\,\cdot\,,\mathcal{F})} & \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R.) \\ \downarrow f & & \uparrow f^{\star} \\ D^{1\times p}/(D^{1\times q}\,R) = M & \xrightarrow{\hom_D(\,\cdot\,,\mathcal{F})} & \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R.) \end{array}$$

Block triangular reduction

- Theorem (Cluzeau-Q. 08): Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying $R\,P = Q\,R$.

If the left D-modules

$$\ker_D(.P), \quad \mathrm{coim}_D(.P) := D^{1\times p}/\ker_D(.P),$$
$$\ker_D(.Q), \quad \mathrm{coim}_D(.Q) := D^{1\times q}/\ker_D(.Q),$$

are free of rank m, $p-m$, l, $q-l$, then there exist

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D),$$

such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & V_1\,R\,W_2 \\ 0 & V_2\,R\,W_2 \end{pmatrix},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times (p-m)}$ and:

$$U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}.$$

Block diagonal decomposition

- Theorem (Cluzeau-Q. 08): Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q \quad \Rightarrow \quad f^2 = f.$$

If the projective left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P) = \ker_D(.(I_p - P)),$$
$$\ker_D(.Q), \quad \mathrm{im}_D(.Q) = \ker_D(.(I_q - Q)),$$

are free of rank m, $p-m$, l, $q-l$, then there exist

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D),$$

such that

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{pmatrix},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times(p-m)}$ and:

$$U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \quad V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}.$$

Block diagonal decomposition

- Corollary (Cluzeau-Q. 08): If D is either

1. D is a principal left ideal domain (e.g., $\mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$),
2. $D = k[x_1, \ldots, x_n]$, where k is a field,
3. $D = A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, where $A = k\{t\}$, $k = \mathbb{R}, \mathbb{C}$, and:

$$\begin{array}{ll} \mathrm{rank}_D(\ker_D(.P)) \geq 2, & \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2, \\ \mathrm{rank}_D(\ker_D(.Q)) \geq 2, & \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2. \end{array} \qquad (\star)$$

4. $D = A_n(k)$ or $B_n(k)$, $\mathrm{char}(k) = 0$, and $(\star)$,

and there exist $P \in D^{p \times p}$ and $Q \in D^{q \times q}$ such that

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q,$$

then there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that:

$$V\,R\,U^{-1} = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}.$$

Differential time-delay examples

- Mounier-Rudolph-Petitot-Fliess 95: $\partial\, f(t) = \dot{f}(t), \delta\, f(t) = f(t-1)$

$$U = \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ 2\,\partial\,(1-\delta^2) & \partial\,\delta\,(\delta^2-1) & -2 \\ -1 & \frac{1}{2}\,\delta & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\delta \end{pmatrix},$$

$$V \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(\delta^2+1) & 0 \end{pmatrix} U^{-1} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- Dubois-Petit-Rouchon 99:

$$\begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \sim \begin{pmatrix} \delta^2-1 & 0 & 0 \\ 0 & 1+\delta^2 & -4\,\partial\,\delta \end{pmatrix}.$$

- Petit-Rouchon 02:

$$\begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \sim \begin{pmatrix} \partial\,(1-\delta)\,(\delta+1) & 0 & 0 \\ 0 & \partial\,(\delta^2+1) & 2\,\alpha\,\partial^2\,\delta \end{pmatrix}.$$

PD examples

- Rotational isentropic flow (Courant-Hilbert): $4\,(u^2 - c^2)\,\alpha^2 = 1$

$$\begin{pmatrix} u\,\rho\,\partial_x & c^2\,\partial_x & 0 \\ 0 & c^2\,\partial_y & u\,\rho\,\partial_x \\ \rho\,\partial_x & u\,\partial_x & \rho\,\partial_y \end{pmatrix} \sim \begin{pmatrix} \partial_x - 2\,\alpha\,c\,\partial_y & 0 & 0 \\ 0 & \partial_x + 2\,\alpha\,c\,\partial_y & 0 \\ 0 & 0 & \partial_x \end{pmatrix}$$

- Dirac equations for a massless particle:

$$\begin{pmatrix} \partial_4 & 0 & -i\,\partial_3 & -i\,\partial_1 - \partial_2 \\ 0 & \partial_4 & -i\,\partial_1 + d_2 & i\,\partial_3 \\ i\,\partial_3 & i\,\partial_1 + \partial_2 & -\partial_4 & 0 \\ i\,\partial_1 - \partial_2 & -i\,\partial_3 & 0 & -\partial_4 \end{pmatrix} \sim \begin{pmatrix} i\,\partial_3 - \partial_4 & -i\,\partial_1 - \partial_2 & 0 & 0 \\ i\,\partial_1 - \partial_2 & i\,\partial_3 + \partial_4 & 0 & 0 \\ 0 & 0 & i\,\partial_3 + \partial_4 & i\,\partial_1 + \partial_2 \\ 0 & 0 & i\,\partial_1 - \partial_2 & -i\,\partial_3 + \partial_4 \end{pmatrix}$$

- Coupled heat equations: $((a_1 - b_2)^2 + 4\,a_2\,b_1)\,\alpha^2 - 1 = 0$

$$\begin{pmatrix} \partial_t - k\,\partial_x^2 - a_1 & -b_1 \\ -a_2 & \partial_t - k\,\partial_x^2 - b_2 \end{pmatrix} \sim \begin{pmatrix} \partial_t - k\,\partial_x^2 - \frac{(a_1 + b_2)}{2} + \frac{1}{2\,\alpha} & 0 \\ 0 & \partial_t - k\,\partial_x^2 - \frac{(a_1 + b_2)}{2} - \frac{1}{2\,\alpha} \end{pmatrix}$$

- Wave/Cauchy-Riemann/Beltrami eqs, electrical line. . .

Serre's reduction

- Theorem (Boudellioua-Q. 10): Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $\Lambda \in D^{q\times(q-r)}$ such that

$$\exists\, U \in \mathrm{GL}_{p+q-r}(D) : \ (R \quad -\Lambda)\, U = (I_q \quad 0),$$

i.e., $\mathrm{coker}_D(.(R \quad -\Lambda))$ is free of rank $p-r$. Then, we have

$$M = D^{1\times p}/(D^{1\times q}\, R) \cong D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2),$$

where $U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}$, $Q_1 \in D^{p\times(p-r)}$ and $Q_2 \in D^{(q-r)\times(p-r)}$.

The converse result holds. The results depend only on:

$$\rho(\Lambda) \in \mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) \cong D^{q\times(q-r)}/(R\, D^{p\times(q-r)}).$$

- Corollary: $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p-r)} \mid Q_2\, \zeta = 0\}$.

System equivalence

- Theorem (Boudellioua-Q. 10):
  1. If $\Lambda \in D^{q \times (q-r)}$ admits a left inverse, then the left D-module $\ker_D(.Q_1)$ is stably free of rank r:
  $$\ker_D(.Q_1) \oplus D^{1 \times (p-r)} \cong D^{1 \times p}.$$
  2. If the left D-module $\ker_D(.Q_1)$ is free, then $\exists\, Q_3 \in D^{p \times r}$ such that $W := (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$. Then, we have:
  $$V^{-1}\, R\, W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}, \quad V := (R\, Q_3 \quad \Lambda) \in \mathrm{GL}_q(D).$$
  The converse of this result holds.

Ring conditions

- Corollary (Boudellioua-Q. 10): If $P = (R \quad -\Lambda) \in D^{q \times (p+r)}$ admits a right inverse, i.e., $P\,S = I_q$, and

1. D is a principal left ideal domain (e.g., $\mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$),
2. $D = k[x_1, \ldots, x_n]$, where k is a field,
3. $D = A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, where $A = k\{t\}$, $k = \mathbb{R}, \mathbb{C}$, and $p - r \geq 2$,
4. $D = A_n(k)$ or $B_n(k)$, $\mathrm{char}(k) = 0$, and $p - r \geq 2$,

$$\Rightarrow \ \exists\, U \in \mathrm{GL}_{p+q-r}(D) : \ (R \quad -\Lambda)\, U = (I_q \quad 0).$$

- Corollary (Boudellioua-Q. 10): If $\Lambda \in D^{q \times (q-r)}$ admit a left inverse and D is either 1, 2 or 3 and 4 with $r \geq 2$, then there exists $Q_3 \in D^{p \times r}$ such that $W := (Q_3 \quad Q_1) \in \mathrm{GL}_p(D)$ and:

$$V^{-1}\, R\, W = \mathrm{diag}(I_r, Q_2), \quad V := (R\, Q_3 \quad \Lambda) \in \mathrm{GL}_q(D).$$

Serre's reduction based on holonomy

• Theorem (Cluzeau-Q. 10): Let $D = A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, where $A = k\{t\}$, $k = \mathbb{R}, \mathbb{C}$, and $R \in D^{q\times p}$ be a full row rank matrix.

1. If $p - q \geq 1$, then there exists $Q \in D^{1\times(p-q+1)}$ such that:

$$M \cong D^{1\times(p-q+1)}/(D\,Q), \quad \ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.).$$

2. If $q \geq 3$, then there exist $W \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ s.t.:

$$V^{-1}\,R\,W = \mathrm{diag}(I_{q-1}, Q). \quad (\star)$$

• Corollary (Cluzeau-Q. 10): Every locally analytic linear OD system with at least 1 input is isomorphic to a locally analytic linear ODE.

If the system is defined by at least 3 equations, then $(\star)$ holds.

String with an interior mass (Fliess et al, 98)

$$(\star)\ \begin{cases} \phi_1(t)+\psi_1(t)-\phi_2(t)-\psi_2(t)=0, \\ \dot{\phi}_1(t)+\dot{\psi}_1(t)+\eta_1\,\phi_1(t)-\eta_1\,\psi_1(t)-\eta_2\,\phi_2(t)+\eta_2\,\psi_2(t)=0, \\ \phi_1(t-2\,h_1)+\psi_1(t)-u(t-h_1)=0, \\ \phi_2(t)+\psi_2(t-2\,h_2)-v(t-h_2)=0. \end{cases}$$

$$V^{-1}\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial_1+\eta_1 & \partial_1-\eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \partial_2^2 & 1 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 1 & \partial_3^2 & 0 & -\partial_3 \end{pmatrix} W$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1+\eta_1+\eta_2 & 2\,\eta_1\,\partial_2 & 2\,\eta_2\,\partial_3 \end{pmatrix},$$

$$(\star) \Leftrightarrow \dot{z}_1(t)+(\eta_1+\eta_2)\,z_1(t)+2\,\eta_1\,z_2(t-h_1)+2\,\eta_2\,z_3(t-h_2)=0.$$

String with an interior mass (Fliess et al, 98)

The unimodular matrices U and V are defined by:

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 & \partial_2 & 0 \\ 0 & -1 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\partial_3 \\ 0 & -1 & -1 & 1 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_2 & \partial_2^2 - 1 & 0 \\ 0 & -\partial_3 & -\partial_3 & \partial_3 & 0 & \partial_3^2 - 1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_2^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_2^2\,(\partial_1 - \eta_1 + \eta_2) - \partial_1 - \eta_1 & 1 & -\partial_1 + \eta_1 - \eta_2 & 2\,\eta_2 \end{pmatrix}.$$

A model of a two reflector antenna (Mounier et al, 97)

$$R = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & 0 & 0 & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial + \frac{K_2}{T_e} & -\frac{K_c}{T_e}\,\delta & -\frac{K_c}{T_e}\,\delta & -\frac{K_p}{T_e}\,\delta \end{pmatrix}$$

$$R \sim \mathrm{diag}(I_3, S, S, S),$$

$$S = ((T_e\,\partial + K_2)\,\partial \quad (K_p + 2\,K_c)\,(K_c - K_p)\,\delta).$$

A model of a two reflector antenna (Mounier et al, 97)

$$V^{-1}\,R\,W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (T_e\,\partial + K_2)\,\partial & (K_p + 2\,K_c)\,(K_c - K_p)\,\delta \end{pmatrix}$$

A model of a two reflector antenna (Mounier et al, 97)

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{T_e\,\partial + K_2}{K_1\,T_e} & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 & 0 & 0 \\ -K_1^{-1} & 0 & 0 & T_e\,\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 & 0 & 0 \\ 0 & -K_1^{-1} & 0 & 0 & 0 & T_e\,\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_1\,T_e & 0 \\ 0 & 0 & -K_1^{-1} & 0 & 0 & 0 & 0 & T_e\,\partial & 0 \\ 0 & 0 & 0 & 0 & T_e\,(K_p + K_c) & 0 & -K_c\,T_e & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & T_e\,(K_p + K_c) & 0 & -K_c\,T_e \\ 0 & 0 & 0 & 0 & -K_c\,T_e & 0 & -K_c\,T_e & 0 & T_e\,(K_p + K_c) \end{pmatrix}$$

Maple packages

- OreModules ⇒ Jacobson (basis computation)
- OreModules ⇒ QuillenSuslin (basis computation)
- OreModules ⇒ Stafford (basis computation)
- OreModules ⇒ OreMorphisms (factorization, reduction and decomposition)
- OreModules ⇒ Serre (Serre's reduction) (in development)

- The philosophy of OreModules and OreMorphisms have largely been extended in the `homalg` package (Barakat) of GAP 4.

⇒ Future implementation of these packages in `homalg`.

⇒ Efficient computations (e.g., Singular, Macaulay2, Maple).