On the Polyhedral Set-Invariance Conditions and Stabilization for Time-Delay Systems

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26 november 2010

Outline

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Introduction and context

- Networked control systems $\rightarrow$ variable delay
- Constrained control $\rightarrow$ Predictive control
- Predictive control:
    - On-line optimization procedure
    - No stability guarantee
    - Invariant set as terminal set of constraints

Initial approach

Overview (ECC 2009, IFAC TDS 2009, IFAC TDS 2010, ACC 2010, CIFA 2010):

- Constrained control for time-delay systems
- Invariant sets
  - Extended state-space framework
  - Classical stabilization techniques
- Classical unconstrained stabilization
  - Extended state-space framework - Classical Lyapunov approach
  - Non-extended state-space framework - Lyapunov-Krasovskii approach

Disadvantages:

- High complexity of the invariant sets in the extended state-space
- Few alternative methods in the literature:
  - Dambrine (1995)
  - Goubet-Bartholomeus (1997)
  - Hennet (1998)
  - Vassilaki (1999)

Initial approach

- Continuous linear system with input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t-h),$$

  where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.
- Degree of uncertainty: $h = dT_e - \epsilon$, with sampling time T_e.
- Discrete model:

$$x_{k+1} = Ax_k + Bu_{k-d} - \Delta(u_{k-d} - u_{k-d+1})$$

  where $u(k)$ is piecewise constant between simpling periods and:

$$A = e^{A_c T_e}, \quad B = \int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta, \quad \Delta = \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau$$

- $\Delta \rightarrow$ exponential function in terms of the uncertainty ϵ.

Initial approach

Objective:

- Robust stability of LTI systems with time-variable delay
- Design a control law which regulates the system state while robustly satisfying a set of constraints:

$$Cx_k + Du_k \leq W$$

where $C \in \mathbb{R}^{r \times n}$ and $W \in \mathbb{R}^r$.

Initial approach

- Extended model:

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k$$

where $A_\Delta \in \mathbb{R}^{n+d \times n+d}$ and $B \in \mathbb{R}^{n+d \times m}$, with:

$$\xi_k^T = \begin{bmatrix} x_k \\ u_{k-d} \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix} ; \; A_\Delta = \begin{bmatrix} A & B-\Delta & \Delta & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ & & & \ddots & \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} ; \; B_\Delta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}$$

- Polytopic model with $n+1$ extreme realizations $\Delta \in Co\{\Delta_0, \dots, \Delta_n\}$
- Global polytopic model in an extended state space:

$$\begin{aligned} \xi_{k+1} &= A_\Delta \xi_k + B_\Delta u_k \\ A_\Delta &\in \Omega \\ \Omega &= \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \dots A_{\Delta_n}\} \end{aligned}$$

Initial approach

- Stabilization:
  - Extended state-space framework:
    Classical LMI-based design: $u(k) = K\xi(k)$, where $K \in \mathbb{R}^{m\times n+d}$.
  - Non-extended state-space framework:
    Lyapunov-Krasovskii LMI-based design: $u(k) = Kx(k)$, where $K \in \mathbb{R}^{m\times n}$.
- Invariant sets:
  - Extended state-space framework:

$$\Gamma\xi(k) + Du(k) \leq \mathbb{W}$$

where $\Gamma \in \mathbb{R}^{n+d\times w}$.

Mathematical Background - Minkowski Addition

- Minkowski addition:
  For two arbitrary sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$

$$\mathcal{A} \oplus \mathcal{B} := \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}.$$

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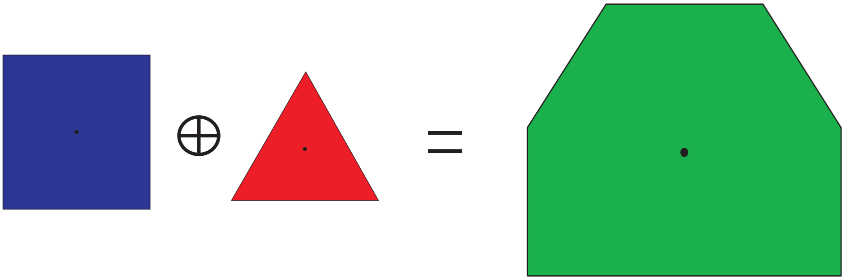
- Example:

$$\mathcal{A} = \{0, 1\},$$
$$\mathcal{B} = \{3, 4\},$$

$$\mathcal{A} \oplus \mathcal{B} = \{3, 4, 5\}.$$

- Example:



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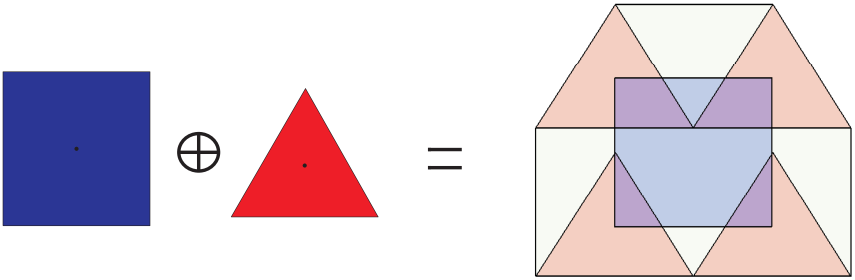
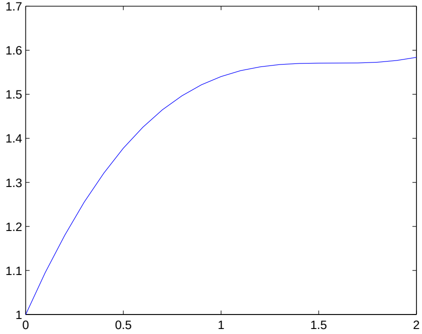


Mathematical Background - Set Dynamics and maps

- Classical maps:
  For $f : \mathbb{R}^n \to \mathbb{R}^n$ we have $y = f(x)$, where $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

- Classical maps:
  For $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have $y = f(x)$, where $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.
  - Example:
    $f(x) = x + \cos(x), \ \forall x \in \mathbb{R}$



Mathematical Background - Set Dynamics and maps

- Set Dynamics or Mappings:
  For $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have $\Phi(\mathcal{S}) \subseteq \mathbb{R}^n$ and $\mathcal{S} \subseteq \mathbb{R}^n$.

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- Examples:
  - Set scaling:
    For a set $\mathcal{A} \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}_+$, $\alpha\mathcal{A} := \{\alpha x \mid x \in \mathcal{A}\}$.
  - Linear set operation:
    For an arbitrary matrix $A \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subseteq \mathbb{R}^n$:

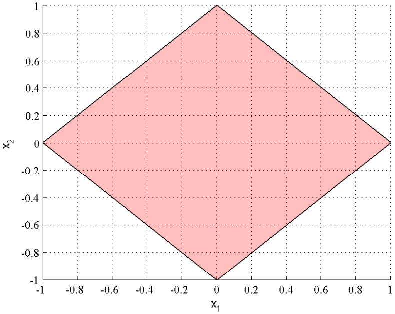
$$A\mathcal{S} = \{y \in \mathbb{R}^n | y = Ax,\, x \in \mathcal{S}\}.$$

  - Linear set iterates:

$$\Phi(\mathcal{S}, k) = A^k\mathcal{S},\ \forall k \in \mathbb{Z}_+.$$

Mathematical Background - Set scaling

- Example: Set scaling.
  $\mathcal{A}$ is 1-norm unit circle in $\mathbb{R}^2$ and $\alpha = 0.5$.
  Set scaling $\alpha\mathcal{A}$:



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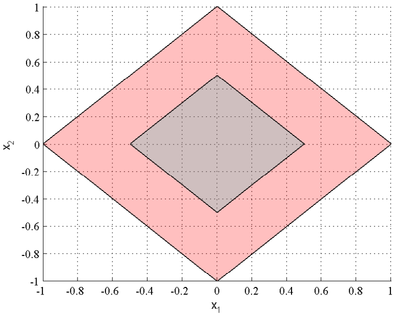


- Example: Compute the mapping:

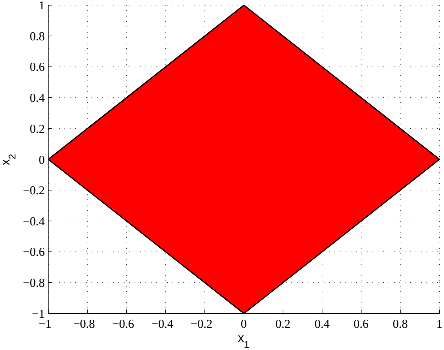
$$\Phi(\mathcal{S}, k) = A^k \mathcal{S},\ \forall k \in \mathbb{Z}_+,$$

where $\mathcal{S}$ is a 1-norm unit circle in $\mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$ is a contractive rotation matrix:

$$A = \rho \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

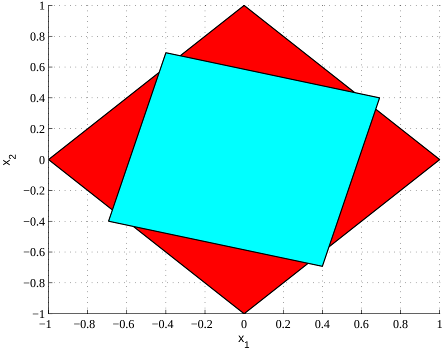
with $\rho \in \mathbb{R}_{[0,1)}$.

For $\theta = \frac{\pi}{6}$ and $\varepsilon = 0.8$.
For $k = 0$, the mapping $\Phi(\mathcal{S}, k) = \mathcal{S}$ is:



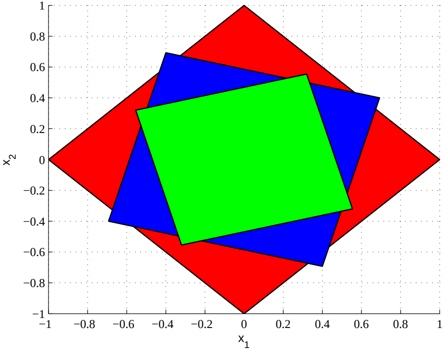
Mathematical Background - Set Dynamics

For $\theta = \frac{\pi}{6}$ and $\varepsilon = 0.8$.
For $k = 1$, the mapping $\Phi(\mathcal{S}, k) = A\mathcal{S}$ is:



Mathematical Background - Set Dynamics

For $\theta = \frac{\pi}{6}$ and $\varepsilon = 0.8$.
For $k = 2$, the mapping $\Phi(\mathcal{S}, k) = A^2\mathcal{S}$ is:



Mathematical Background - Hausdorff Distance

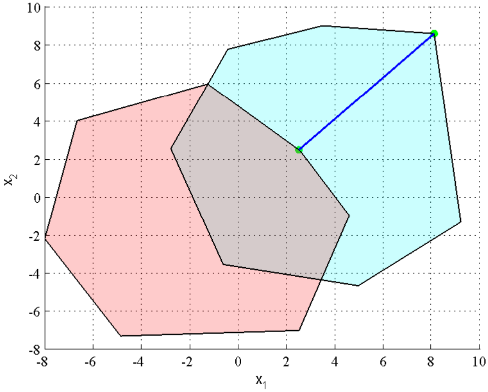
- Hausdorff Distance: Distance between two sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$ given by:

$$d_H(\mathcal{A}, \mathcal{B}) = \max \left(\max_{x \in \mathcal{A}} \min_{y \in \mathcal{B}} d(x, y), \ \max_{x \in \mathcal{B}} \min_{y \in \mathcal{A}} d(x, y) \right),$$

where $d(x, y)$ is the Euclidean distance between the points x and y in $\mathbb{R}^n$. In practical, this distance can be obtained by solving a QP problem, in terms of distance between hyperplanes and extreme points.

Mathematical Background - Hausdorff Distance

Example: Find the Haudorff distance between the two sets:



$$d_H = 8.26$$

Outline

Classical Results Concerning Polyhedral Set Invariance

Definition (Set invariance)

Let $\varepsilon \in \mathbb{R}_{[0,1)}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called *contractive* with respect to system

$$x(k+1) = f(x(k))$$

if

$$f(\mathcal{P}) \subseteq \varepsilon \mathcal{P}.$$

For $\varepsilon = 1$, $\mathcal{P}$ is called an *invariant* set with respect to the dynamics.

Classical Results Concerning Polyhedral Set Invariance

Proposition (Bitsoris 1988)

The convex polyhedral set:

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\},$$

with $F \in \mathbb{R}^{r \times n}$, $w \in \mathbb{R}^r$, *is an invariant set with respect to*

$$x(k+1) = Ax(k),$$

with $A \in \mathbb{R}^{n \times n}$, *if there exists a matrix* $H \in \mathbb{R}^{r \times r}$ *with nonnegative elements such that:*

$$FA = HF$$

and

$$Hw \leq w.$$

Classical Results Concerning Polyhedral Set Invariance

Definition ((Blanchini 1995) Minkowski functions)

Consider a convex and compact polyhedral set containing the origin:

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq \mathbb{1}\},$$

with $F \in \mathbb{R}^{r \times n}$, $w \in \mathbb{R}^r$. The polyhedral function associated to $\mathcal{P}$ is called a Minkowski function:

$$V(x) = \max_{j \in \mathbb{Z}_{[1,r]}} \{\max\{\{(Fx)_j\}, 0\}\}.$$

where $\{(Fx)_j\}$ denotes the j^{th} element of Fx. This function can be seen as a vector infinity-norm (Kiendl et al. 1992):

$$V(x) = \|\max\{Fx, 0\}\|_\infty.$$

Classical Results Concerning Polyhedral Set Invariance

Remark

The Minkowski function of a set $\mathcal{P}$ can be used as polyhedral Lyapunov candidate.

Remark

Consider $\varepsilon \in \mathbb{R}_{[0,1)}$. One of the statements of the Lyapunov stability theorem is:

$$V(x(k+1)) - \varepsilon V(x(k)) \leq 0$$

If $\varepsilon = 1$ the function $V(x)$ is called a **weak Lyapunov function**. *Although the existence of a weak Lyapunov function does not imply global asymptotic stability, it induces invariant sets.*

Outline

The $\mathcal{D}$-Invariance

- Delay-difference equation of the form:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i),$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n \times n}$, for all $i \in \mathbb{Z}_{[0,d]}$. Initial conditions satisfy $x(-i) \in \mathbb{R}^n$, for all $i \in \mathbb{Z}_{[0,d]}$.

The $\mathcal{D}$-Invariance

Definition ($\mathcal{D}$*(elay)* Invariance)

Let $\varepsilon \in \mathbb{R}_{[0,1)}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ containing the origin is called $\mathcal{D}$-contractive set with respect to the system if:

$$\bigoplus_{i=0}^{d} A_i \mathcal{P} \subseteq \varepsilon \mathcal{P}.$$

When $\varepsilon = 1$, $\mathcal{P}$ is called a $\mathcal{D}$-invariant set with respect to the dynamics.

The $\mathcal{D}$-Invariance

- Some properties:

The $\mathcal{D}$-Invariance

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  - If $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant then $\alpha\mathcal{P}$ is $\mathcal{D}$-invariant for any $\alpha \in \mathbb{R}_{>0}$.

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- Some properties:
  - If $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant then $\alpha\mathcal{P}$ is $\mathcal{D}$-invariant for any $\alpha \in \mathbb{R}_{>0}$.
  - Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ be two $\mathcal{D}$-invariant sets for a given dynamics. Then $\mathcal{A} \cap \mathcal{B}$ is a $\mathcal{D}$-invariant set for the same dynamical system.

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  - Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ be two $\mathcal{D}$-invariant sets for a given dynamics. Then $\mathcal{A} \cap \mathcal{B}$ is a $\mathcal{D}$-invariant set for the same dynamical system.
  - Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a convex set containing the origin. If $\mathcal{P}$ is $\mathcal{D}$-invariant with respect to:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i),$$

then $\mathcal{P}$ is positive invariant with respect to the time invariant linear dynamics:

$$\begin{aligned} x(k+1) &= A_0 x(k); \\ x(k+1) &= A_1 x(k); \\ &\vdots \\ x(k+1) &= A_d x(k). \end{aligned}$$

Equivalently, $A_0\mathcal{P} \subseteq \mathcal{P}$, $A_1\mathcal{P} \subseteq \mathcal{P}$, ..., $A_d\mathcal{P} \subseteq \mathcal{P}$.

The $\mathcal{D}$-Invariance

- Some properties:
  - Given a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ for the system:

$$x(k+1) = A_0 x(k) + \cdots + A_d x(k-d)$$

then $\mathcal{P}$ is $\mathcal{D}$-invariant for:

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$$x(k+1) = A_d x(k) + \cdots + A_0 x(k-d).$$

  - Given a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ for the system:

$$x(k+1) = A_0 x(k) + A_1 x(k-1),$$

then $\mathcal{P}$ is $\mathcal{D}$-invariant for:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x(k-2); \\ &\vdots \\ x(k+1) &= A_0 x(k) + A_1 x(k-d). \end{aligned}$$

The $\mathcal{D}$-Invariance

Example:

- Set $\mathcal{P}$: ∞-norm unit circle.
- Delay-difference equation of the form:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i),$$

where $n = 2$ and $d = 1$, with the matrices:

$$A_0 = \begin{bmatrix} 0.2 & -0.34 \\ 0.34 & 0.2 \end{bmatrix};\ A_1 = \begin{bmatrix} 0.24 & -0.17 \\ 0.17 & 0.24 \end{bmatrix}.$$

The $\mathcal{D}$-Invariance

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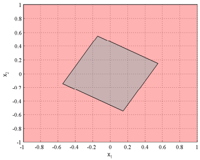
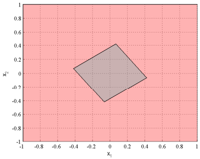
Is the set $\mathcal{P}$ a $\mathcal{D}$-invariant set with respect to the dynamics?

The $\mathcal{D}$-Invariance

$\mathcal{P}$ (red) and $A_0\mathcal{P}$ (black)



$\mathcal{P}$ (red) and $A_1\mathcal{P}$ (black)



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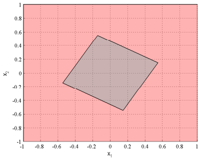
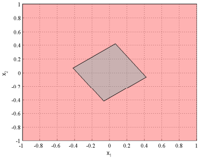
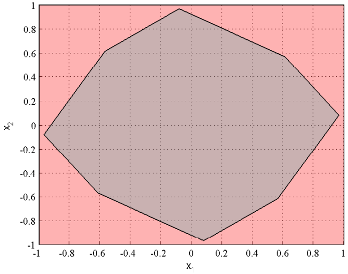


$\mathcal{P}$ (red) and $A_1\mathcal{P}$ (black)



$$A_0\mathcal{P} \oplus A_1\mathcal{P} \subseteq \mathcal{P}$$



$\mathcal{D}$-Invariance Algebraic Conditions

Theorem (Hennet and Tarbouriech (1998))

Let $\mathcal{P}$ be a polyhedral set in $\mathbb{R}^n$ containing the origin in its interior, i.e. there exists a $F \in \mathbb{R}^{r \times n}$ such that:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\}$$

is $\mathcal{D}$-contractive with respect to the system if and only if there exist the matrices $H_i \in \mathbb{R}^{r \times r}$ for $i \in \mathbb{Z}_{[0,d]}$ with non-negative elements such that:

$$FA_i = H_i F$$

and

$$\left(\sum_{i=0}^{d} H_i \right) \mathbb{1} \leq \varepsilon \mathbb{1}.$$

When $\varepsilon = 1$, $\mathcal{P}$ is called a $\mathcal{D}$-invariant set.

Cyclic $\mathcal{D}$-Invariance Conditions

Definition ($\mathcal{D}$-Invariance)

Consider the sets $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i \in \mathbb{Z}_{[0,d]}$. The sets $\mathcal{P}_i \subseteq \mathbb{R}^n$ containing the origin are called $\mathcal{D}$-invariant sets with respect to the system if:

$$
\begin{gathered}
A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \oplus \cdots \oplus A_d\mathcal{P}_d \subseteq \mathcal{P}_0; \\
A_0\mathcal{P}_d \oplus A_1\mathcal{P}_0 \oplus \cdots \oplus A_d\mathcal{P}_{d-1} \subseteq \mathcal{P}_d; \\
\vdots \\
A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_2 \oplus \cdots \oplus A_d\mathcal{P}_0 \subseteq \mathcal{P}_1.
\end{gathered}
$$

Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

Theorem

Let $\mathcal{P}_i$ be polyhedral sets in $\mathbb{R}^n$ containing the origin in its interior, for $i \in \mathbb{Z}_{[0,d]}$, i.e. there exist $F_i \in \mathbb{R}^{r \times n}$ such that:

$$\mathcal{P}_i = \{x \in \mathbb{R}^n \mid F_i x \leq \mathbb{1}\}$$

are $\mathcal{D}$-invariant with respect to the system if and only if there exist the matrices $H_{ij} \in \mathbb{R}^{r \times r}$ for $i, j \in \mathbb{Z}_{[0,d]}$ with non-negative elements such that:

$$F_i A_j = H_{ij} F_i$$

and

$$\left(\sum_{j=0}^{d} H_{0j}\right)\mathbb{1} \leq \mathbb{1}; \quad \cdots ; \quad \left(\sum_{j=0}^{d} H_{dj}\right)\mathbb{1} \leq \mathbb{1}.$$

$\mathcal{D}$-Invariance Algebraic Conditions

Theorem (Lyapunov-Razumikhin Theorem)

Consider the Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that there exist the radially unbounded functions* $\phi(\cdot), \omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous and non-decreasing with* $\phi(0) = \omega(0) = 0$ *and* $\varepsilon \in \mathbb{R}_{[0,1)}$. *Denote:*

$$\mathbf{x}(k)^\top = \begin{bmatrix} x(k)^\top & x(k-1)^\top & \dots & x(k-d)^\top \end{bmatrix}^\top \in (\mathbb{R}^n)^{d+1}.$$

Consider the function $\tilde{V} : (\mathbb{R}^n)^{d+1} \to \mathbb{R}^n$ *with :*

$$\tilde{V}(\mathbf{x}(k)) \triangleq \max_{i \in \mathbb{Z}_{[0,d]}} \{V(x(k-i))\}.$$

If the following hold:

(i) $\phi(\|x\|) \leq V(x) \leq \omega(\|x\|),\ \forall x \in \mathbb{R}^n$,

(ii) $V(x(k+1)) - \varepsilon \tilde{V}(\mathbf{x}(k)) \leq 0,\ \forall k \in \mathbb{Z}_+,\ \forall \mathbf{x}(0) \in (\mathbb{R}^n)^{d+1}$,

then the system is globally asymptotically stable.

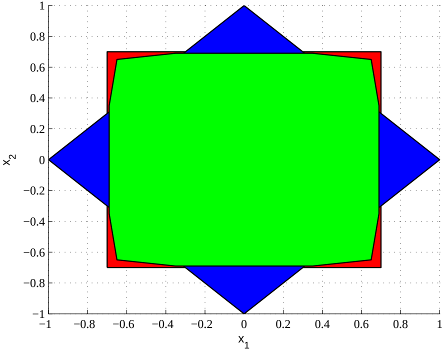
Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

- Example:
  Consider the sets $\mathcal{P}_0, \mathcal{P}_0$ are ∞-norm unit circles and $\mathcal{P}_1, \mathcal{P}_3$ are 1-norm unit circles. Consider also the system matrices:

$$A_0 = \begin{bmatrix} 0 & 0.25 \\ 0.25 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.3 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0.25 \\ 0.25 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.04 \end{bmatrix}$$

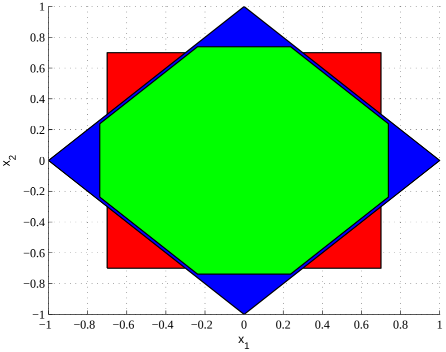
Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

$$A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_1 \oplus A_2\mathcal{P}_2 \oplus A_3\mathcal{P}_3 \subseteq \mathcal{P}_0$$



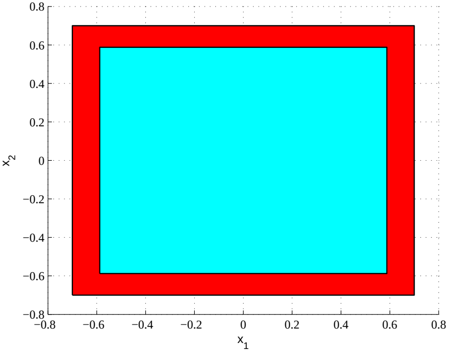
Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

$$A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_2 \oplus A_2\mathcal{P}_3 \oplus A_3\mathcal{P}_0 \subseteq \mathcal{P}_1$$



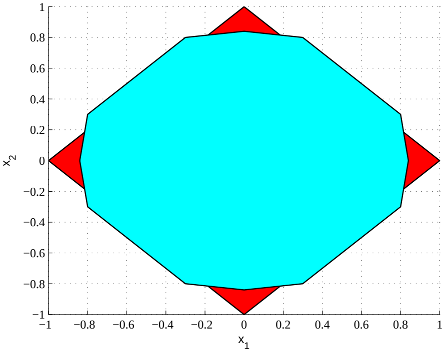
Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

$$A_0\mathcal{P}_0 \oplus A_1\mathcal{P}_0 \oplus A_2\mathcal{P}_0 \oplus A_3\mathcal{P}_0 \subseteq \mathcal{P}_0$$



Cyclic $\mathcal{D}$-Invariance Algebraic Conditions

$$A_0\mathcal{P}_1 \oplus A_1\mathcal{P}_1 \oplus A_2\mathcal{P}_1 \oplus A_3\mathcal{P}_1 \not\subseteq \mathcal{P}_0$$



The $\mathcal{D}$-Invariance verification methods

- Minkowski addition based tests
  - Verify the inclusion by the direct application of the $\mathcal{D}$-invariance definition
  - Based on vertex enumeration
  - Computationally expensive
  - Intractable for high order Euclidean space dimensions
- Feasibility based tests
  - $\mathcal{D}$-invariant algebraic conditions
  - Half-space representation
  - LP feasibility problem

The $\mathcal{D}$-Invariance verification methods

- $\mathcal{D}$-invariance feasibility test:

Theorem

The polyhedral set $\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \leq \mathbb{1}\}$ is $\mathcal{D}$-invariant with respect to the system $x(k+1) = \sum_{i=0}^{d} A_i x(k-i)$, if there exists the vector $\bar{h} \in \mathbb{R}^{dr^2}$ with nonnegative elements obtained by:

$$\min_{\bar{h}} \quad \varepsilon$$

$$\text{subject to:} \quad \begin{cases} A_{eq}\bar{h} = b_{eq} \\ A_{in}\bar{h} \leq b_{in} \\ \bar{h} \geq 0 \\ 0 \leq \varepsilon \leq 1 \end{cases}$$

If $\varepsilon < 1$ the set $\mathcal{P}$ is $\mathcal{D}$-contractive.

The $\mathcal{D}$-Invariance verification methods

Proof:
Taking the previous result:

$$FA_i = H_i F, \quad \text{and} \quad \left(\sum_{i=0}^{d} H_i \right) \mathbb{1} \leq \varepsilon \mathbb{1}.$$

The $\mathcal{D}$-Invariance verification methods

Proof:
The matrices H_i are present in both equalities and inequalities:

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where $\bar{h}$ is a vector formed by the elements of H_i, for $i \in \mathbb{Z}_{[0,d]}$ and ε.

The $\mathcal{D}$-Invariance stabilization

- Discrete time-delay system:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i) + \sum_{i=0}^{d} B_i u(k-i),$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^r$ is the control input at the time $k \in \mathbb{Z}_+$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, for all $i \in \mathbb{Z}_{[0,d]}$.

- State feedback control law:

$$u(k) = Kx(k),$$

with $K \in \mathbb{R}^{m\times n}$.

- Closed loop:

$$x(k+1) = \sum_{i=0}^{d} (A_i + B_i K_i) x(k-i).$$

The $\mathcal{D}$-Invariance stabilization

Proposition

The polyhedral set of constraints $\mathcal{P} = \{x \in \mathbb{R}^n \mid Fx \le w\}$, with $F \in \mathbb{R}^{r \times n}$, is $\mathcal{D}$-invariant with respect to the system

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i) + \sum_{i=0}^{d} B_i u(k-i),$$

and the control law $u(k) = K_i x(k)$ is a solution to the stabilization of the system if and only if there exist the matrices $H_i \in \mathbb{R}^{r \times r}$ for $i \in \mathbb{Z}_{[0,d]}$, $K \in \mathbb{R}^{m \times n}$ and $\varepsilon \in \mathbb{R}_{[0,1)}$ such that:

$$F(A_i + B_i K_i) = H_i F,$$

$$\left(\sum_{i=0}^{d} H_i \right) w \le \varepsilon w.$$

The $\mathcal{D}$-Invariance stabilization

Proof: The same principle of the $\mathcal{D}$-invariant verification.

$\mathcal{D}$-invariant Sets Construction

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  - Is this set the biggest one?
- Delay-difference equation:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-i),$$

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- Set of state constraints $\mathcal{D} \subseteq \mathbb{R}^n$, i.e. $x(k) \in \mathcal{D}$ for $k \in \mathbb{R}_+$. The initial conditions satisfy $x(-i) \in \mathcal{D}$, for all $i \in \mathbb{Z}_{[0,d]}$.

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- Obtain an invariant set $\mathcal{P} \subseteq \mathbb{R}^n$ with respect to the dynamics such that $\mathcal{P} \subseteq \mathcal{D}$.

$\mathcal{D}$-invariant Sets Construction

- Mappings:
  1. Set dynamics in direct time:

$$\begin{aligned} \Phi: \quad & ComC(\mathbb{R}^n) \rightarrow ComC(\mathbb{R}^n) \\ & \Phi(\mathcal{D}) = \bigoplus_{i=0}^{d} A_i \mathcal{D} \end{aligned}$$

$\rightarrow$ Non-monotone mapping
$\rightarrow$ Non-monotone Hausdorff distance

$\mathcal{D}$-invariant Sets Construction

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     $\rightarrow$ Non-monotone mapping
     $\rightarrow$ Non-monotone Hausdorff distance
  2. Convex hull between the set dynamics and $\mathcal{D}$:

$$\Psi : \quad ComC(\mathbb{R}^n) \rightarrow ComC(\mathbb{R}^n)$$
$$\Psi(\mathcal{D}) = Co(\mathcal{D}, \bigoplus_{i=0}^{d} A_i \mathcal{D}) = Co(\mathcal{D}, \Phi(\mathcal{D}))$$

     where $ComC(\mathbb{R}^n)$ denotes compact and convex sets in $\mathbb{R}^n$ and $Co(\mathcal{A}, \mathcal{B})$ is the convex hull between $\mathcal{A}$ and $\mathcal{B}$, subsets of $\mathbb{R}^n$.
     $\rightarrow$ Monotone mapping
     $\rightarrow$ Monotone Hausdorff distance

$\mathcal{D}$-invariant Sets Construction - Iterative procedure

- k set iterates can be defined as:

$$\Phi^k(\mathcal{D}) = \Phi(\Phi^{k-1}(\mathcal{D})), k \geq 0 \text{ with } \Phi^0(\mathcal{D}) = \mathcal{D},$$
$$\Psi^k(\mathcal{D}) = \Psi(\Psi^{k-1}(\mathcal{D})), k \geq 0 \text{ with } \Psi^0(\mathcal{D}) = \mathcal{D}.$$

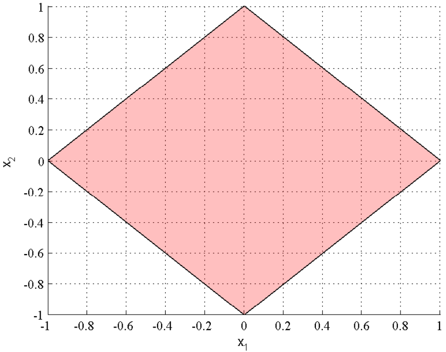
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

- Basic iterative procedure to obtain $\mathcal{P}$, a $\mathcal{D}$-invariant set:
  1) $i = 1$;
  2) Calculate: $P_i = \Psi^i(\mathcal{D})$;
  3) Calculate the Hausdorff distance: $d_H(P_i, P_{i-1})$;
  4) If $d_H = 0$, stop. Otherwise do $i = i + 1$ and go to step 2.
     The obtained set $\mathcal{P}$ is a $\mathcal{D}$-invariant set.
  5) Find α such that $\alpha\mathcal{P} \subseteq \mathcal{D}$.

- Example:
  $\mathcal{D}$ is a 1-norm unit circle in $\mathbb{R}^2$ and $A_i \in \mathbb{R}^{2\times 2}$, with $i = 0, 1$, are contractive rotation matrices.
  For A_0, $\rho = 0.5$ and $\theta = \frac{\pi}{6}$.
  For A_1, $\rho = 0.45$ and $\theta = \frac{\pi}{4}$.

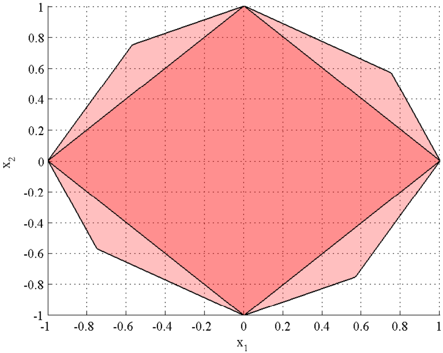
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

Set iterate $\Psi^0(\mathcal{D})$:



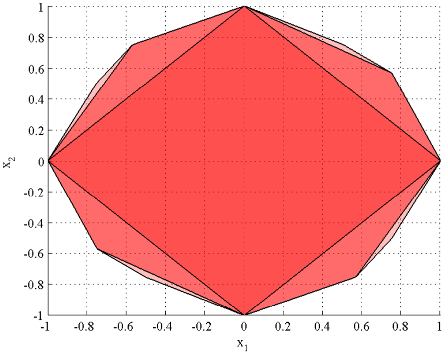
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

Set iterate $\Psi^1(\mathcal{D})$:



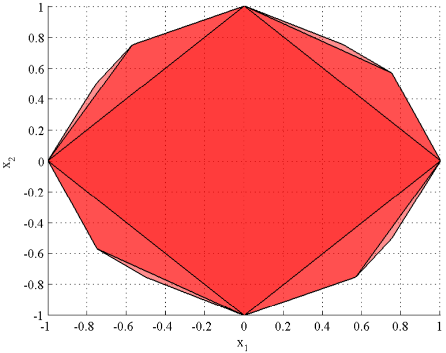
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

Set iterate $\Psi^2(\mathcal{D})$:



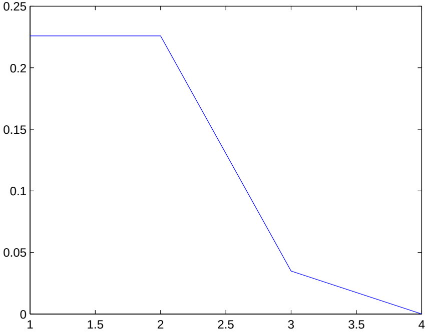
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

Set iterate $\Psi^3(\mathcal{D})$:



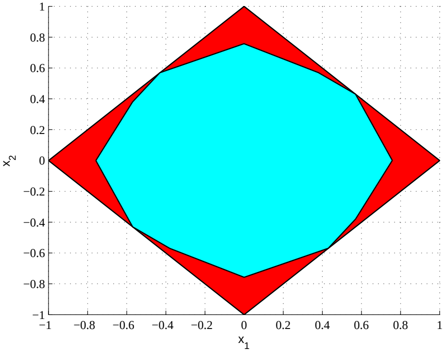
$\mathcal{D}$-invariant Sets Construction - Iterative procedure

Hausdorff distance $d_H(P_i, P_{i-1})$:



$\mathcal{D}$-invariant Sets Construction - Iterative procedure

- Find α such that $\alpha\mathcal{D} \subseteq \mathcal{P}$: $\alpha = 0.7579$.



$\mathcal{D}$-invariant Sets Construction - Iterative procedure

- Is this set the maximal one? (Probably not)
- Find the maximal one
- Sketch of the algorithm:
  1) Take a point $p \in \mathcal{D}$ but $p \notin \mathcal{P}$;
  2) Test its invariance ;
  3) If it is invariant make $Co(p, \mathcal{P})$ and go to 1, calculate the d_H between the sets; otherwise take another p closer to $\mathcal{P}$ and go to 2;
  4) The algorithm stops when d_H is smaller than a given precision.

Outline

Conclusions and Perspectives

Conclusions:

- Constrained states for time-delay systems
- Invariant sets in the non augmented state-space $\rightarrow$ $\mathcal{D}$-invariance
- Mathematical background
- Classical results on polyhedral set invariance
- The $\mathcal{D}$-invariance
  - Minkowski addition based conditions
  - Algebraic based conditions
  - Stabilization
  - $\mathcal{D}$-invariant sets algorithmic construction

Perspectives:

- Is there a delay-dependent approach?
- Practical application (position control of a DC motor)

Thanks a lot!
Questions and comments...