Une approche géométrique pour les systèmes non linéaires à retards; application à l'équivalence linéaire

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Rappel de la problématique non linéaire

exemple sans retard

$$\dot{x} = \begin{pmatrix} x_2 \\ 1 \end{pmatrix} u$$

On calcule toujours le noyau à gauche

$$\begin{bmatrix} 1 & -x_2 \end{bmatrix} = dx_1 - x_2 dx_2 = d(x_1 - \frac{x_2^2}{2})$$

D'où, un état "non commandable" ☺

$$d(x_1 - \frac{x_2^2}{2}) / dt = 0$$

Rappel de la problématique non linéaire

exemple avec retard

$$\dot{x}(t) = \begin{pmatrix} x_2(t-1) \\ 1 \end{pmatrix} u(t)$$

On calcule toujours le noyau à gauche

$$\begin{bmatrix} 1 & -x_2(t-1) \end{bmatrix} = dx_1(t) - x_2(t-1)dx_2(t)$$

Non intégrable ☹

Et tout devient “commandable”

Rappel de la problématique non linéaire

exemple avec retard

$$\dot{x}(t) = \begin{pmatrix} x_2(t-1)u(t-1) \\ u(t) \end{pmatrix}$$

On calcule toujours le noyau à gauche

$$\begin{bmatrix} 1 & -x_2(t-1)\delta \end{bmatrix} = dx_1(t) - x_2(t-1)dx_2(t-1)$$

intégrable ☺

D'où le nouvel état "non commandable"

$$d(x_1(t) - \frac{x_2^2(t-1)}{2})/dt = 0$$

Outils dans le cas sans retard

Lemme de Poincaré

Théorême de Frobenius

Pas d'équivalent dans le cas avec retard

Presentation Outline

- Bibliography
- The Class of Systems – Notations
- Bicausal Change of Coordinates
- Definition of the Extended Lie Bracket
- The Accessibility Submodules
- The linear equivalence problem
- An Example

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The Class of Systems

$$\dot{x}(t) = F(x(t), \cdots x(t - sD)) + \sum_{j=0}^{s} G_j(x(t), \cdots x(t - sD))u(t - jD)$$

x Є R^n, u Є R, F(x(t),…x(t-s)), Gj(x(t),…x(t-s)) *smooth loc. around x_0*

sD = maximum delay

Assumption: The delay D is unknown and constant=>

$$\Sigma : \quad \dot{x}(t) \quad = \quad F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t - j)$$

$$\mathbf{x}_{[s]}^T = (x^T(t), \cdots x^T(t - s))$$

Notations (1/2)

- $\mathcal{K}$: field of meromorphic functions of a *finite* number of symbols in

$\{x(t-i), u(t-i), \dot{u}(t-i), \ldots, u^{(k)}(t-i), i, k \in I\!N\}$

$$f(t) = x(t)\,x^2(t-2) \in \mathcal{K}.$$

- $\mathcal{E} = \text{span}_{\mathcal{K}}\{dx(t-i), du(t-i), d\dot{u}(t-i), \ldots, du^{(k)}(t-i), i, k \in I\!N\}$.

- $d : \mathcal{K} \to \mathcal{E}$ differential operator

$$df(t) = 2x(t)x(t-2)dx(t-2) + x^2(t-2)dx(t) = 2x(t)x(t-2)\delta^2 dx + x^2(t-2)dx.$$

- δ : backward time-shift operator:

$$\delta\, a(t)\, df(t) \quad = \quad a(t-1)\delta df(t) = a(t-1)df(t-1), \qquad a(\cdot), f(\cdot) \in \mathcal{K}$$

- $\deg(\cdot)$ is the polynomial degree in δ of its argument.

Notations (2/2)

- $\mathcal{K}(\delta]$ is the (left) ring of polynomials in δ with coefficients in $\mathcal{K}$.

$\mathcal{K}(\delta] \ni \alpha(\delta] = \alpha_0(t) + \alpha_1(t)\,\delta + \cdots + \alpha_{r_\alpha}(t)\,\delta^{r_\alpha}, \quad \alpha_i \in \mathcal{K}$, where $r_\alpha = \deg(\alpha(\delta])$.

Addition: $\alpha(\delta] + \beta(\delta] = \sum_{i=0}^{\max\{r_\alpha, r_\beta\}} (\alpha_i(t) + \beta_i(t))\delta^i$

Multiplication $\alpha(\delta]\beta(\delta] = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(t)\,\beta_j(t-i)\delta^{i+j}$.

[Xia, Marquez-Martinez, Zagalak, Moog, 2002].

- $\Delta = \mathrm{span}_{\mathcal{K}(\delta]}\{r_1(\cdot,\delta), \ldots, r_s(\cdot,\delta)\}$

- $\mathbf{x}_{[s]}^T = (x^T(t), \cdots x^T(t-s))$ and $\mathbf{x}_{[s]}(-p) = (x^T(t-p), \cdots x^T(t-s-p))$;

Nonlinear Time Delay Systems

$$\Sigma: \quad \dot{x}(t) \quad = \quad F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j)$$

The associated differential form representation is

$$\Sigma_L: \quad d\dot{x} \quad = \quad f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)dx + g(\mathbf{x}_{[s]}, \delta)du$$

$$f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \quad = \quad \sum_{i=0}^{s} \frac{\partial F(\mathbf{x}_{[s]})}{\partial x(t-i)}\delta^i + \sum_{j=0}^{s} u(t-j) \sum_{i=0}^{s} \frac{\partial G_j(\mathbf{x}_{[s]})}{\partial x(t-i)}\delta^i$$

$$g(\mathbf{x}_{[s]}, \delta) \quad = \quad \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})\delta^j.$$

Bicausal Changes of Coordinates

Definition 1: $z = \phi(\mathbf{x}_{[\alpha]}), \phi \in \mathcal{K}^n$ is a *bicausal* change of coordinates for

$$\Sigma : \quad \dot{x}(t) = F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j)$$

if $\exists\ \ell \in I\!N$ integer, and $\phi^{-1}(\mathbf{z}_{[\ell]}) \in \mathcal{K}^n$ s.t. $x(t) = \phi^{-1}(\mathbf{z}_{[\ell]})$.

$$\dot{x} = F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j)$$

$z(t) = \phi(\mathbf{x}_{[\alpha]})$

$$\dot{z} = \tilde{F}(\mathbf{z}_{[\gamma]}) + \sum_{j=0}^{\gamma} \tilde{G}_j(\mathbf{z}_{[\gamma]})u(t-j)$$

$x(t) = \phi^{-1}(\mathbf{z}_{[\ell]})$.

Bicausal Changes of Coordinates -Properties

$$z(t) = \phi(\mathbf{x}_{[\alpha]}) \longrightarrow dz = T\left(\mathbf{x}_{[\gamma]}, \delta\right) dx$$

- $T\left(\mathbf{x}_{[\gamma]}, \delta\right) = \sum_{i=0}^{\alpha} \frac{\partial \phi(\mathbf{x}_{[\alpha]})}{\partial x(t-i)} \delta^i = \sum_{i=0}^{\alpha} T^i(\mathbf{x}_{[\gamma]}) \delta^i$, unimodular

$$x(t) = \phi^{-1}(\mathbf{z}_{[\ell]}) \longrightarrow dx = T^{-1}\left(\mathbf{z}, \delta\right) dz$$

- $T^{-1}\left(\mathbf{z}, \delta\right) = \sum_{i=0}^{\ell} \frac{\partial \phi^{-1}(\mathbf{z}_{[\ell]})}{\partial z(t-i)} \delta^i = \sum_{i=0}^{\ell} \bar{T}^i(\mathbf{z}) \delta^i$ satisfies

$$T^0(\mathbf{x})|_{\phi^{-1}(\mathbf{z})} \bar{T}^0(\mathbf{z}) = I$$

$$\sum_{i=1}^{k} T^i(\mathbf{x})|_{\phi^{-1}(\mathbf{z})} \bar{T}^{k-i}(\mathbf{z}(-i)) = 0, \qquad \forall k \geq 1.$$

Linear Equivalence - Problem Formulation

$$\Sigma: \quad \dot{x}(t) \quad = \quad F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j)$$

x $\in R^n$, $u \in R$, s=max. delay, F(.), G_j(.) smooth $F(x_0)=0$.

Find if it exists z=Φ(x) bicausal s.t. in the new variables

$$\dot{z} = \sum_{j=1}^{l} A_j z(t-j) + \sum_{j=1}^{l} B_j u(t-j)$$

for some suitable matrices A_j, B_j

Nonlinear Time delay Systems

Problems

1-when a unimodular matrix represents the differential of a bicausal change of coordinates?

2-how can we compute the bicausal change of coordinates which solves the linear equiv. Problem?

Delay free case: key tool $\mathcal{G}_{n-1}$:=span$\{g, ad_f g, \ldots, ad_f^{n-1} g\}$

rank $\mathcal{G}_{n-1}$ =n + nilpotency condition + cond. on $ad_f^n g$

3- Who is $ad_f^i g$ for delay systems ?

The Extended Lie Bracket operator

Definition 2: $\ldots)\delta^j.$

$$[r_1^k(\cdot), r_2^l(\quad \ldots \quad i)) \frac{\partial}{\partial x(t-j)}$$

with

$$[r_1^k(\cdot), r_2^l(\cdot \quad \ldots \quad r_2^{l-i}(\mathbf{x}(-i)).$$

$$\left[\begin{pmatrix} r_1^k \\ r_1^{k-1}(-1) \\ \vdots \\ r_1^0(-k) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} r_2^l \\ r_2^{l-1}(-1) \\ \vdots \\ r_2^{l-k}(-l+k) \\ \vdots \\ \\ r_2^0(-l) \end{pmatrix} \right]$$

The Exte ... omponents over $R^{(i+1}$

Nilpotent Submodules

$$\Delta = \text{span}_{\mathcal{K}(\delta]}\{r_1(\mathbf{x},\delta),\cdots,r_j(\mathbf{x},\delta)\}$$

$$r_l(\mathbf{x},\delta) = \sum_{t=0}^{s} r_l^t(\mathbf{x})\delta^t$$

Given $\mathbf{x}^0 = (x^0(t)^T, x^0(t-1)^T, \cdots x^0(t-\gamma)^T)^T$

Definition 3: Δ is nonsingular locally around $\mathbf{x}^0$ if

$$\text{rank}(\Delta(\mathbf{x})) = j, \quad \forall \mathbf{x} \in \mathcal{U}_0$$

Definition 4: Δ nonsingular locally around $\mathbf{x}^0$, is nilpotent if

$$[r_l^t(\cdot), r_i^p(\cdot)]_{E,2s} = 0, \qquad \forall t \leq p \in [0, 2s].$$

Nilpotency - Interpretation

$$
\left\{
\begin{array}{cccccccccc}
r^0(\mathbf{x}) & r^1(\mathbf{x}) & r^2(\mathbf{x}) & \cdots & r^s(\mathbf{x}) & 0 & 0 & \cdots & 0 & \cdots \\
0 & r^0(\mathbf{x}(-1)) & r^1(\mathbf{x}(-1)) & \cdots & r^{s-1}(\mathbf{x}(-1)) & r^s(\mathbf{x}(-1)) & 0 & \cdots & \vdots & \cdots \\
0 & 0 & r^0(\mathbf{x}(-2)) & \cdots & r^{s-2}(\mathbf{x}(-2)) & r^{s-1}(\mathbf{x}(-2)) & r^s(\mathbf{x}(-2)) & \ddots & \cdots & \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & & 0 & \cdots \\
0 & & \cdots & 0 & r^0(\mathbf{x}(-s)) & r^1(\mathbf{x}(-s)) & r^2(\mathbf{x}(-s)) & \cdots & r^s(\mathbf{x}(-s)) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \cdots
\end{array}
\right\}
$$

Consider $R_j^k(\mathbf{x}) = \sum_{i=0}^{k} [r_j^{k-i}(\mathbf{x}(-i))]^T \frac{\partial}{\partial x(t-i)}$

Verifying $\frac{\partial R_i^l}{\partial x_e} R_j^k - \frac{\partial R_j^k}{\partial x_e} R_i^l = 0$ for k,l=0,…,2s, ensures

that it is satisfied for any index

Bicausal Change of Coordinates

Theorem 1: Given $T(\mathbf{x},\delta) = [r_1(\mathbf{x},\delta),\cdots,r_n(\mathbf{x},\delta)] \in \mathcal{K}^{n\times n}(\delta]$

of full rank locally around $\mathbf{x}^0$.

Then $\exists z = \phi(\mathbf{x})$ bicausal with $dz = T^{-1}(\mathbf{x},\delta)dx$ iff

i) $T(\mathbf{x},\delta)$ is unimodular

ii)$\Delta = \mathrm{span}_{\mathcal{K}(\delta]}\{r_1(\mathbf{x},\delta),\cdots,r_n(\mathbf{x},\delta)\}$ nilpotent: $\forall l,j \in [1,n]$

$[r_j^i(\mathbf{x}), r_l^k(\mathbf{x})]_{Ei} = 0$

(equivalently $[r_j^i(\mathbf{x}), r_l^k(\mathbf{x})]_{E0} = 0$) $\forall i \leq k \in [0, 2s]$

Accessibility submodules (1/2)

Set $g_1(\mathbf{x}_{[s]}, \delta) = g(\mathbf{x}_{[s]}, \delta)$

$$g_k(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) = f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) g_{k-1}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) - \dot{g}_{k-1}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta),\ k \geq 2$$

Definition 5: Accessibility submodules

$$\mathcal{R}_i := \text{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x}, \delta) \cdots g_i(\mathbf{x}, \mathbf{u}, \delta)\},\ i \geq 1$$

Proposition 1: If $g_{i+1}(\mathbf{x}, \mathbf{u}, \delta) \in \mathcal{R}_i$

then $\forall j \geq 1,\ g_{i+j}(\mathbf{x}, \mathbf{u}, \delta) \in \mathcal{R}_i$.

Accessibility submodules (2/2)

Proposition 2: Under $z=\phi(\mathbf{x}_{[\alpha]})$ bicausal, with $dz=T(\mathbf{x},\delta)dx$

$$\tilde{g}_j(\mathbf{z},\mathbf{u},\delta)=[T(\mathbf{x},\delta)g_j(\mathbf{x},\mathbf{u},\delta)]_{\mathbf{x}=\phi^{-1}(\mathbf{z})}\,, \qquad 1\leq j\leq n$$

$$[\tilde{g}_p^k(\mathbf{z},\mathbf{u}),\tilde{g}_t^l(\mathbf{z},\mathbf{u})]_{Ei}=\left(\Gamma^{l-i}(\mathbf{x})[g_p^k(\mathbf{x},\mathbf{u}),g_t^l(\mathbf{x},\mathbf{u})]_{Ei}\right)_{|\mathbf{x}=\phi^{-1}(\mathbf{z})}\,,\ k\leq l$$

$$\Gamma^{l-i}(\mathbf{x})=\begin{pmatrix} T^0(\mathbf{x}) & T^1(\mathbf{x}) & \cdots & T^{l-i}(\mathbf{x}) \\ 0 & T^0(\mathbf{x}(-1)) & \cdots & T^{l-i-1}(\mathbf{x}(-1)) \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & T^0(\mathbf{x}(-l+i)) \end{pmatrix}.$$

Properties of Strongly Accessible LTDS

1) $g_i(\cdot,\delta) = g_i(\delta) = A^{i-1}(\delta)B(\delta),$

z=Φ(x) bicausal preserves the independence on u

2) $R_n(\mathbf{x}) = (B(\delta), A(\delta)B(\delta), \cdots A^{n-1}(\delta)B(\delta))$

z=Φ(x) bicausal preserves strong accessibility that is full rank and unimodularity of the accessibility matrix

3) $g_{n+1}(\delta) = A^n(\delta)B(\delta) = \sum_{i=1}^{n} g_i(\delta)c_i(\delta)$

Under z=Φ(x) bicausal g_{n+1} is still a linear combination of the g_i's through the same coefficients $c_i(\delta)$

4) z=Φ(x) bicausal preserves the nilpotency of

$$\mathcal{R}_n(\mathbf{x}) = \mathrm{span}_{\mathcal{K}(\delta]}\{B(\delta), A(\delta)B(\delta), \cdots A^{n-1}(\delta)B(\delta)\}$$

Linear Equivalence of Time delay systems

Theorem 2: System (1) is equivalent, under a bicausal change of coordinates, to a linear strongly accessible delay system if and only if

a) for $1 \leq i \leq n$, $g_i(\cdot) := g_i(\mathbf{x}, \delta)$

b) $R_n(\mathbf{x}) = (g_1(\mathbf{x}, \delta), \cdots g_n(\mathbf{x}, \delta)) = T^{-1}(\mathbf{x}, \delta)$ is unimodular

c) $g_{n+1}(\cdot) \in \text{span}_{R(\delta)}\{g_1(\mathbf{x}, \delta), \cdots g_n(\mathbf{x}, \delta)\}$

d) for $i, j \in [1, n]$ and $r \leq k \in [0, 2\bar{s}]$, the following relation are satisfied

$$[g_i^r(\mathbf{x}), g_j^k(\mathbf{x})]_{E,2\bar{s}} = 0$$

with $g_l(\mathbf{x}, \delta) = g_l^0(\mathbf{x}) + g_l^1(\mathbf{x})\delta + \cdots g_l^{\bar{s}}(\mathbf{x})\delta^{\bar{s}}$.

An Example (1/2)

Consider the Dynamics

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + 2x_2(t-1)u(t-1)\\ \dot{x}_2(t) &= u(t)\end{aligned}$$

$$g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2\,x_2(t-1) \\ 0 \end{pmatrix}\delta,\ g_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ g_3 = 0$$

$$g_3 \in \mathrm{span}_{\mathcal{R}(\delta)}\{g_1, g_2\}$$

The accessibility matrix is thus unimodular and given by

$$R(\mathbf{x}) = \begin{pmatrix} 2\,x_2(t-1)\,\delta & 1 \\ 1 & 0 \end{pmatrix}$$

An Example (2/2)

$$\begin{pmatrix} 0 & 1 & 2x_2(t-1) & 0 & & \\ 1 & 0 & 0 & 0 & & \\ & & 0 & 1 & 2x_2(t-2) & 0 \\ & & 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow [g_i^k, g_j^l]_{E0} = 0$$

$$T(\mathbf{x}, \delta) = R^{-1}(\mathbf{x}, \delta) = \begin{pmatrix} 0 & 1 \\ 1 & -2x_2(t-1)\delta \end{pmatrix}$$

is the diff. Repr. of the bicausal change of coord.

$z_1(t)=x_2(t)$

$z_2(t)=x_1(t)-[x_2\ (t-1)]^2$

$$\dot{z}_1(t) = u(t)$$

$$\dot{z}_2(t) = z_1(t)$$

Conclusions

- We have introduced the Extended Lie Bracket to deal with nonlinear time delay systems
- It has been successively used to define the conditions for the equivalence to a linear strongly accessible time delay system
- This operator seems suitable to define the conditions for the solutions of classic control problems such as feedback linearization, equivalence to canonical forms, etc…