Approximation of distributed delays

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LABORATOIRE AMPÈRE

Application of Distributed delay

- Dead-time compensators
- Finite-spectrum assignment (Manitius & Olbrot,1979 Watanabe,1986)
- The modified Smith predictor (Watanabe & Ito 1981 and Raton,1996)
- H_∞ control of dead-time systems (Zhong, 2003)
- Continuous time dead beat control
- ...

Content

Introduction

Banach Algebra $\mathscr{A}$ (Callier & Desoer, 1978)

$$f(t) = \begin{cases} f_a(t) + f_{pa}(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f_a(\cdot) \in \mathscr{L}_1(\mathbb{R}_+), \quad f_{pa}(t) = \sum_{n=0}^{\infty} f_n \delta(t - t_n)$$

Norm over $\mathscr{A}$:

$$\|f\|_{\mathscr{A}} = \|f_a\|_{\mathscr{L}_1} + \sum_{n=0}^{\infty} |f_n|$$

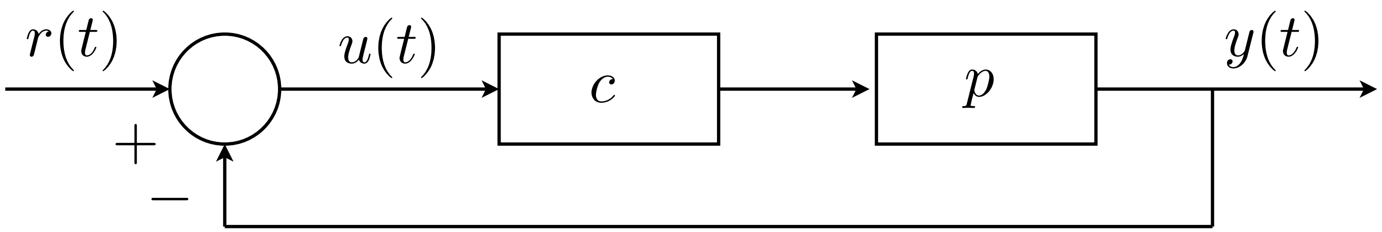
Definition

A convolution system

$$y(t) = (f * u)(t) = \int_0^t f(\tau)u(t-\tau)\mathrm{d}\tau$$

is said to be BIBO stable if $f \in \mathscr{A}$

Example



The plant

$$\hat{p} = \frac{e^{-s}}{s-1}$$

Using the compensator

$$\hat{c}(s) = \frac{2e}{1 + 2\frac{1-e^{-(s-1)}}{s-1}}$$

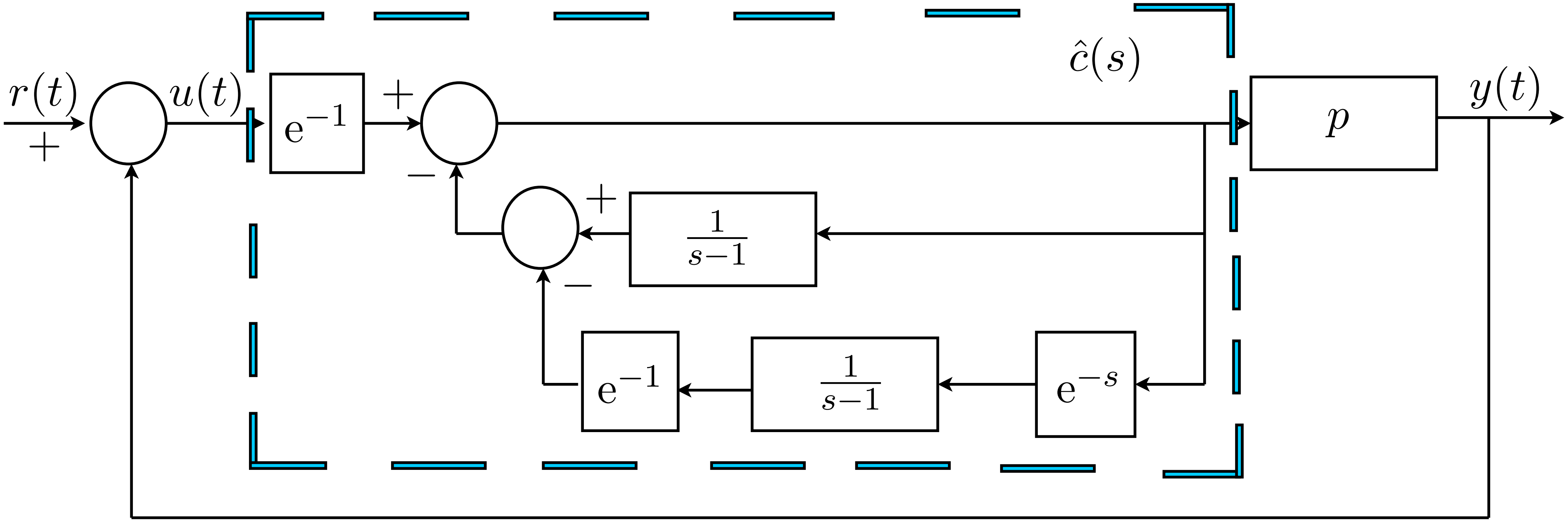
we obtain

$$\hat{y}(s) = \frac{2ee^{-s}}{s+1}\hat{r(s)}$$

A realization of $\hat{c}(s)$ is

$$u(t) = 2er(t) - 2\int_0^1 e^{\tau}u(t-\tau)d\tau - 2ey(t)$$

How to implementation $\hat{c}(s) = \frac{2\mathrm{e}}{1+2\frac{1-\mathrm{e}^{-(s-1)}}{s-1}}$?

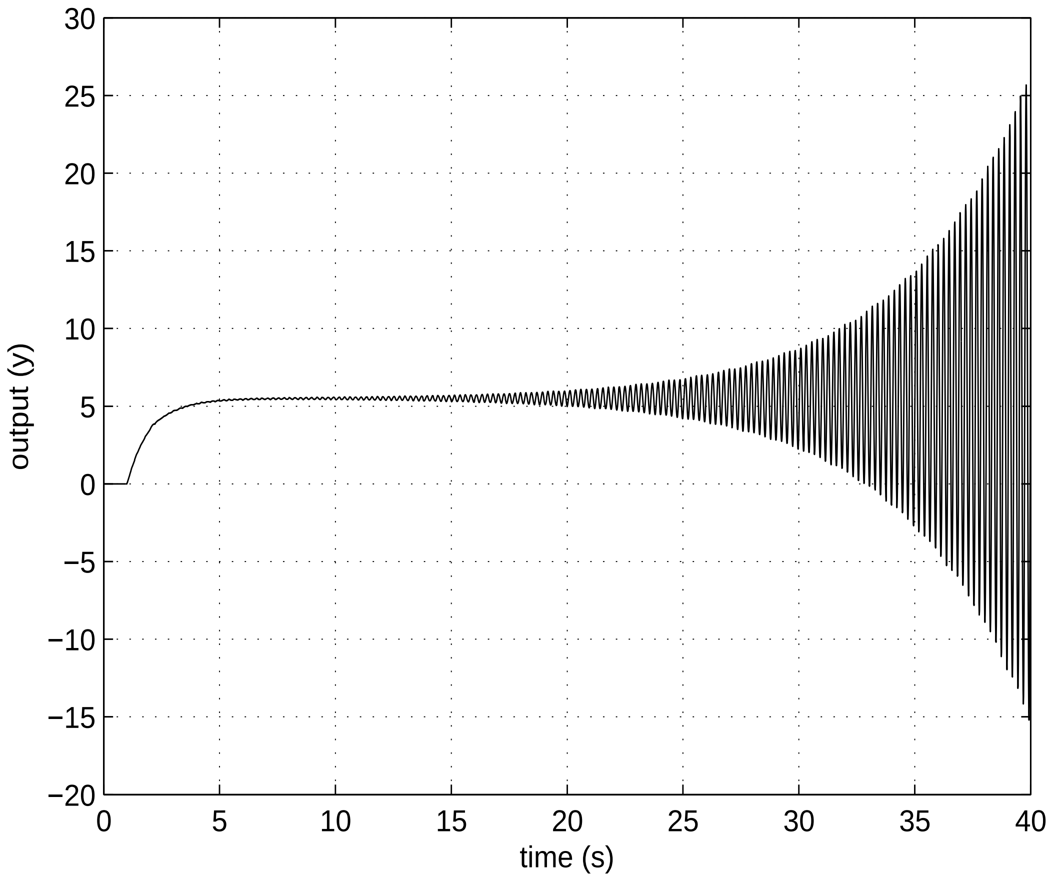


Stability problem ?

$$u(t) = 2\mathrm{e}r(t) - 2\int_0^1 \mathrm{e}^{\tau} u(t-\tau)\mathrm{d}\tau - 2\mathrm{e}y(t)$$

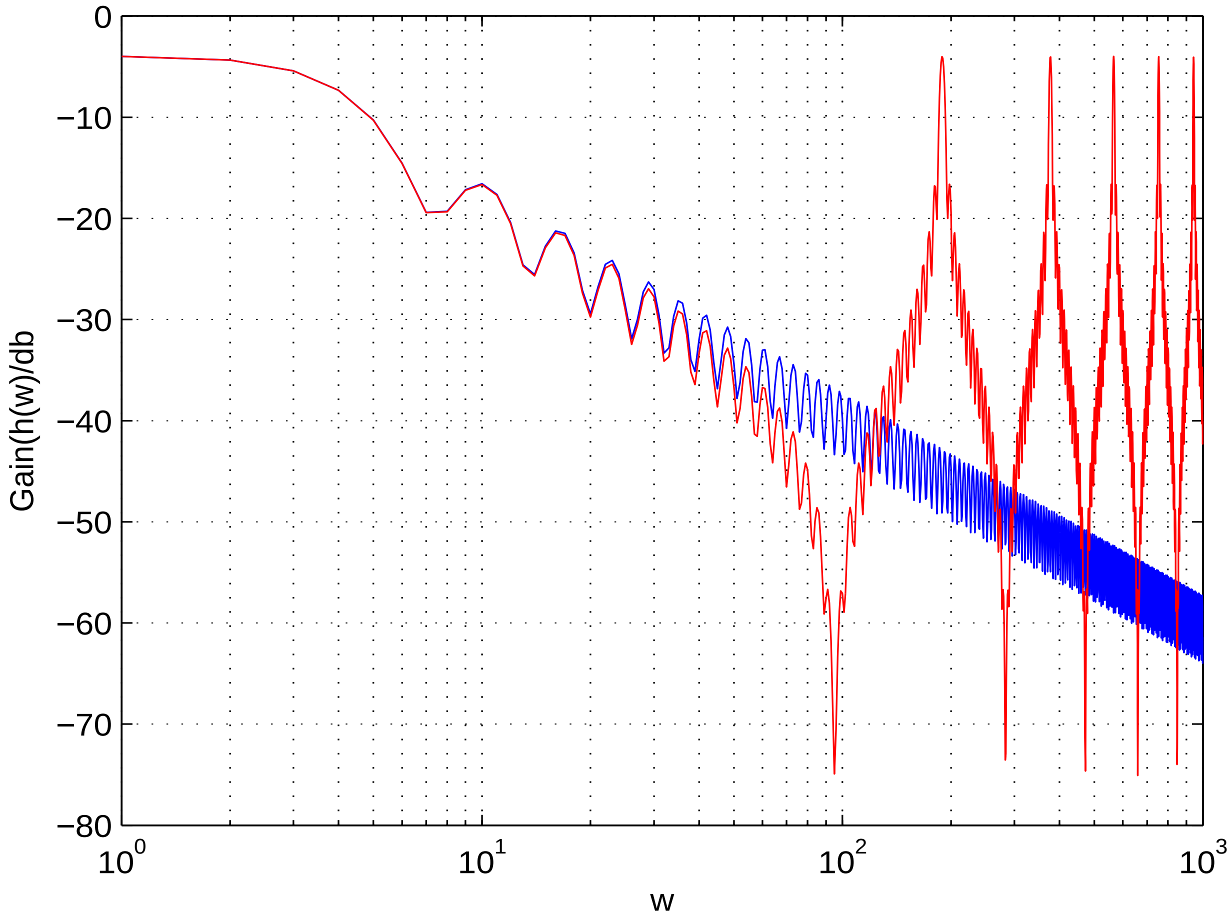
Approximation of the distributed time operator : Newton-Cotes approximation

$$\int_0^1 \mathrm{e}^{\tau} u(t-\tau)\mathrm{d}\tau \approx \frac{1}{\mu}\left[\frac{1}{2}u(t) + \frac{1}{2}\mathrm{e}u(t-1) + \sum_{i=1}^{\mu-1} \mathrm{e}^{\frac{i}{\mu}} u(t-\frac{i}{\mu})\right]$$



In frequency domain

$$\frac{1-\mathrm{e}^{-(s-1)}}{s-1} \text{v.s.} \frac{1}{\mu}\left(\frac{1}{2}+\frac{1}{2}\mathrm{e}\mathrm{e}^{-s}+\sum_{i=1}^{\mu-1}\mathrm{e}^{\frac{i}{\mu}}\mathrm{e}^{-\frac{i}{\mu}s}\right)$$



Problems :

1. Approximation of the distributed time operator

$$y(t) = \int_{\vartheta_1}^{\vartheta_2} f_{\mathbb{I}_{\vartheta_1,\vartheta_2}}(\tau) u(t-\tau)\,\mathrm{d}\tau$$

2. Implementation for control problems in continuous time

Some results in the literature

– Insert a low pass filter (Mirkin,2003)

Adding the filter $\frac{f}{s+f}$ to the convolution and using the Newton-Cotes approximation we can obtain

$$\begin{cases} \dot{z}(t) = -fz(t) + 2f\left\{\mathrm{e}r(t) - \frac{1}{\mu}\left[\frac{1}{2}u(t) + \frac{1}{2}\mathrm{e}u(t-1) + \sum_{i=1}^{\mu-1}\mathrm{e}^{\frac{i}{\mu}}u(t-\frac{i}{\mu})\right] - 2\mathrm{e}y(t)\right\} \\ u(t) = z(t) \end{cases}$$

– Using series and power series expand (Zhong, 2004)

$$v_f(t) = \int_0^h \mathrm{e}^{\tau} u(t-\tau)\mathrm{d}\tau \approx \sum_{i=0}^{N-1} \mathrm{e}^{\frac{h}{N}} u(t - i\frac{h}{N}) * p(t)$$

where $p(t) = 1(t) - 1(t - \frac{h}{N})$

The transfer function form u to v_f is

$$Z_f(s) = \frac{1 - \mathrm{e}^{-s\frac{h}{N}}}{s} \sum_{i=0}^{N-1} \mathrm{e}^{i\frac{h}{N}(s-1)}$$

The hold filter can be expanded as the following series of ϵ :

$$\frac{1 - \mathrm{e}^{-s\frac{h}{N}}}{s} = \frac{1 - \mathrm{e}^{-\frac{h}{N}(s+\epsilon)}}{s+\epsilon} + \frac{1 - \mathrm{e}^{-\frac{h}{N}(s+\epsilon)} - \frac{h}{N}(s+\epsilon)\mathrm{e}^{\frac{h}{N}(s+\epsilon)}}{(s+\epsilon)^2}\epsilon + \cdots$$

For guarantying the static gain let

$$\frac{1 - \mathrm{e}^{-s\frac{h}{N}}}{s} \approx \frac{1 - \mathrm{e}^{-\frac{h}{N}(s+\epsilon)}}{s+\epsilon} \frac{\frac{h}{N}\epsilon}{1 - \mathrm{e}^{-\epsilon\frac{h}{N}}}$$

Distributed delay

$\mathscr{K}(\mathbb{I}_{a,b})$ as the set of real valued functions $g(\cdot)$ of the form

$$g(t) = \begin{cases} g_{\mathbb{I}_{a,b}}(t), & t \in \mathbb{I}_{a,b} \\ 0, & \text{elsewhere} \end{cases}$$

where

$$g_{\mathbb{I}_{a,b}}(t) = \sum_{i \geq 0} \sum_{j \geq 0} c_{ij}\, t^j\, \mathrm{e}^{\lambda_i t}$$

Definition : A distributed delay is a causal convolution system of the form

$$y(t) = (f * u)(t) = \int_0^t f(\tau) u(t - \tau)\, \mathrm{d}\tau$$

where kernel f lies in $\mathscr{K}(\mathbb{I}_{\vartheta_1,\vartheta_2})$, $0 \leq \vartheta_1 < \vartheta_2$.

Laplace transform :

$$\hat{y}(s) = \hat{f}(s)\hat{u}(s), \quad \hat{f}(s) = \int_{\vartheta_1}^{\vartheta_2} f_{\mathbb{I}_{\vartheta_1,\vartheta_2}}(\tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau$$

where $\hat{f}$ is an entire function

Example :

$$f(t) = \begin{cases} \mathrm{e}^t\,, & t \in [0,1] \\ 0\,, & \text{elsewhere} \end{cases}$$

$$\hat{f}(s) = \frac{1 - \mathrm{e}^{-(s-1)}}{s-1}, \quad \hat{f}(1) = 1$$

Elementary distributed delay

$$\theta_\lambda(t) = \begin{cases} \mathrm{e}^{\lambda\, t}\,, & t \in [0, \vartheta] \\ 0\,, & \text{elsewhere} \end{cases}$$

Its Laplace transform is

$$\hat{\theta}_\lambda(s) = \frac{1 - \mathrm{e}^{-(s-\lambda)\vartheta}}{s - \lambda}$$

which is an entire function even in $s = \lambda$ where $\hat{\theta}_\lambda(\lambda) = \vartheta$

The kth derivative $\hat{\theta}_\lambda^{(k)}(s)$ of $\hat{\theta}_\lambda(s)$ is

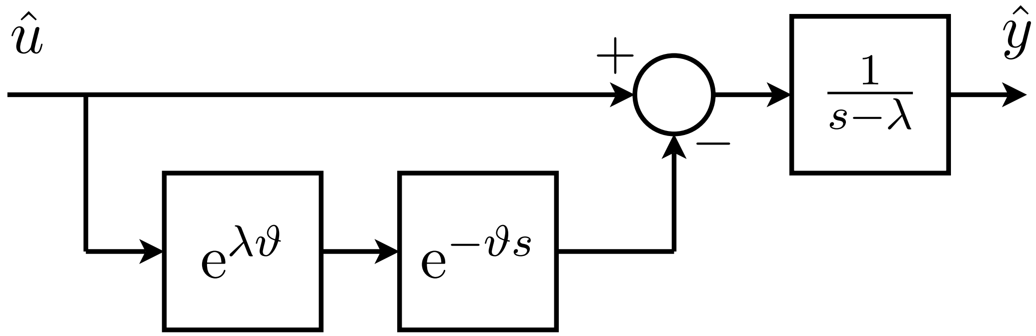
$$\hat{\theta}_\lambda^{(k)}(s) = \int_0^\vartheta (-\tau)^k \mathrm{e}^{-(s-\lambda)\tau} \mathrm{d}\tau$$

$\hat{\theta}_\lambda^{(k)}(s)$ is still distributed delay

Lemma

Any element in distributed delay can be decomposed into a finite sum of Laplace transforms of elementary distributed delays and its successive derivatives.

Implementation of $\hat{\theta}_\lambda(s) = \frac{1-\mathrm{e}^{-(s-\lambda)\vartheta}}{s-\lambda}$



– For $\mathrm{Re}\lambda < 0$: stable implementation.

– For $\mathrm{Re}\lambda \geq 0$: Numerical instability. How to approximate ? $\rightarrow$ using elements distributed delay in "stable form" : $\sum_{i\geq 0} \sum_{j\geq 0} c_{ij}\, t^j\, \mathrm{e}^{\lambda_i t}$ $(\lambda_i < 0)$

Approximation of distributed delay

Classes of approximation

- Lumped systems
- Lumped delayed systems
- Distributed delay in "stable form" : $\mathrm{Re}\lambda < 0$

- Approximation in graph topology over $\mathscr{A}$.

$$f_{app} \in \mathscr{B}(\theta_\lambda, \varepsilon) = \left\{\theta_{\lambda,\mathrm{app}}(t) \in \mathscr{A}, \|\theta_{\lambda,\mathrm{app}}(t) - \theta_\lambda(t)\|_{\mathscr{A}} \leq \varepsilon\right\}$$

  Since $\theta_\lambda \in \mathscr{L}_1$, we approximate it over $\mathscr{L}_1$.

- Approximation by lumped systems
- Approximation by distributed delay in "stable form"

Lemma

Any distributed delay with kernel $\theta_\lambda(\cdot)$ in “unstable form”($\mathrm{Re}\lambda \geq 0$) can be approximated by distributed delays with kernels in “stable form” ($\mathrm{Re}\lambda < 0$) for the graph topology

The Laplace transform of $\theta_{\lambda,\mathrm{app}}(t)$ is

$$\hat{\theta}_{\lambda,\mathrm{app}}(s) = \sum_{k=1}^{n} \xi_k \hat{\theta}_k(s)$$

$\hat{\theta}_k(s)$ in “stable form” ($\mathrm{Re}\lambda < 0$), ξ_k is constant.

Brief Proof and Procedure

The kernel is

$$\theta_\lambda(t) = \begin{cases} e^{\lambda t}, & t \in [0, \vartheta] \\ 0, & \text{elsewhere} \end{cases}$$

- Let $t = -\alpha^{-1}\ln\rho$, we have

$$\psi_\lambda(\rho) = (\alpha\rho)^{-1}\theta_\lambda(-\alpha^{-1}\ln\rho) \quad \rho \in [e^{-\alpha\vartheta}, 1]$$

- Change the variable $\mu = \frac{\rho - e^{-\alpha\vartheta}}{1 - e^{-\alpha\vartheta}}$, we have

$$\Phi_\lambda(\mu) = \theta_\lambda(-\alpha^{-1}\ln((1 - e^{-\alpha\vartheta})\mu) + e^{-\alpha\vartheta})$$

$\mu \in [0, 1]$.

- Using Bernstein polynomials

$$\Phi_{\lambda,app}(\mu) = \sum_{k=0}^{n} C_k^n \Phi_\lambda \left(\frac{k}{n}\right) \mu^k (1-\mu)^{n-k}$$

- Change the variable back from μ to t : $\mu = \frac{e^{-\alpha t} - e^{-\alpha\vartheta}}{1-e^{-\alpha\vartheta}}$ we have

$$\theta_{\lambda,n}(t) = \frac{1}{(1-e^{-\alpha\vartheta})^n} \sum_{k=0}^{n} C_n^k \Phi_\lambda \left(\frac{k}{n}\right) (e^{-\alpha t} - e^{-\alpha\vartheta})^k (1-e^{-\alpha t})^{n-k}$$

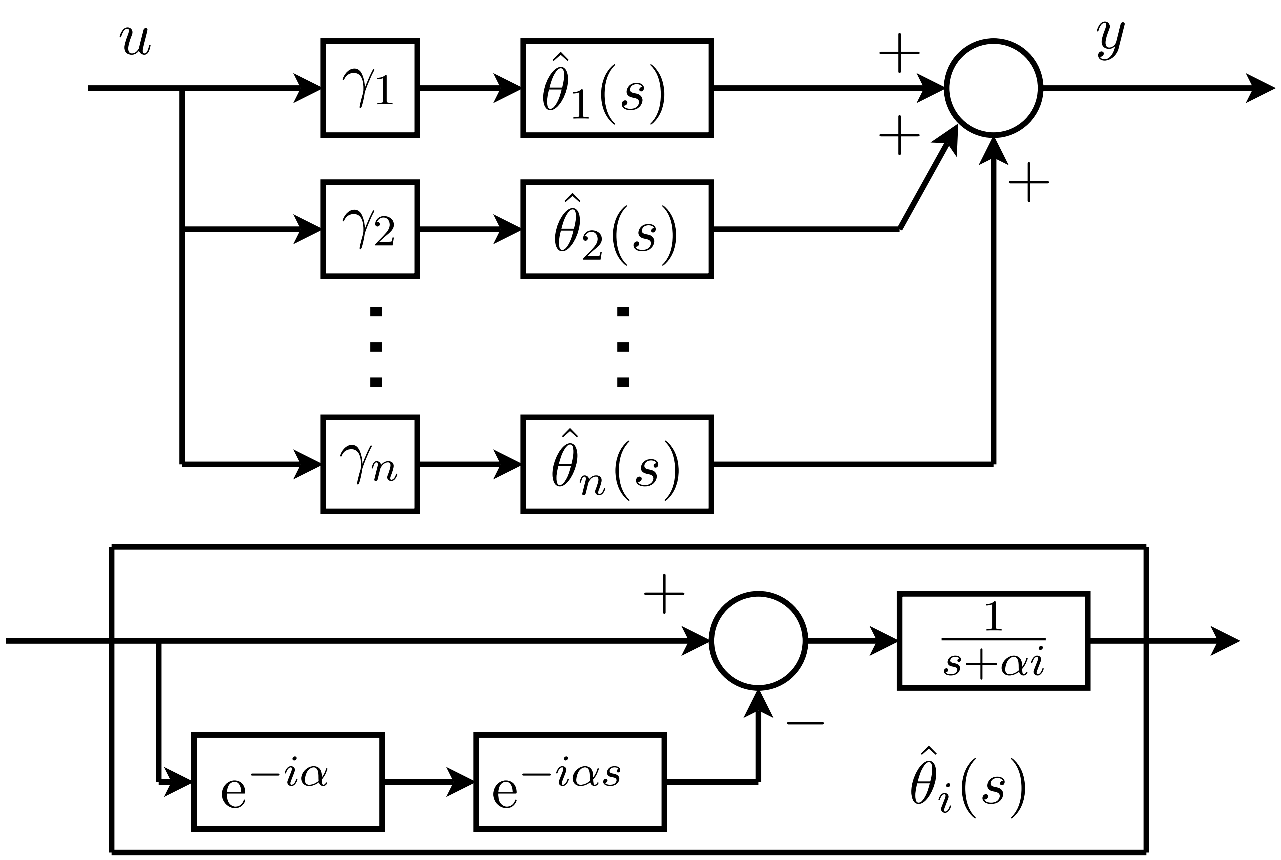
$\theta_{\lambda,n}(t)$ uniformly converges to $\theta_\lambda(\cdot)$ in $\mathscr{L}_1$.

with Laplace transform we have

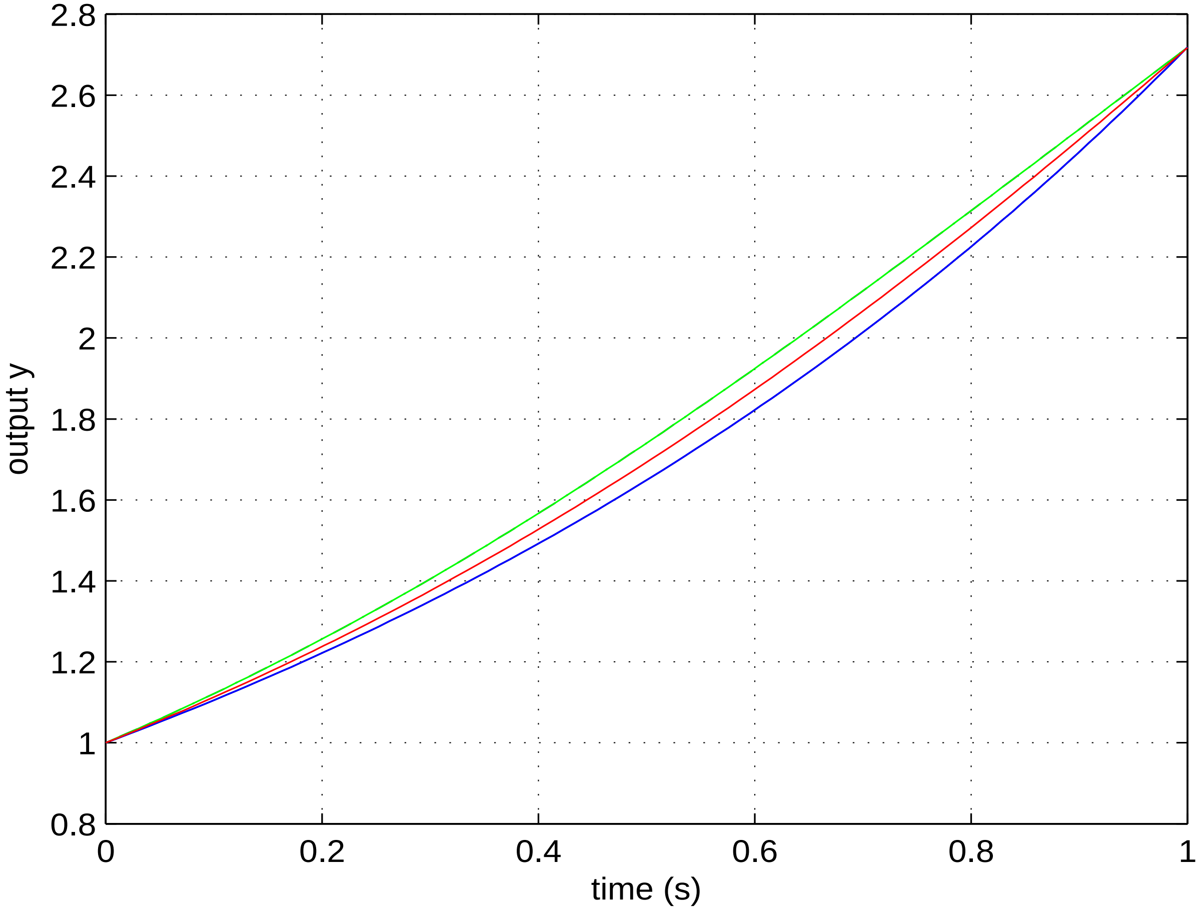
$$\hat{\theta}_{\lambda,\mathrm{app}}(s) = \sum_{k=1}^{n} \gamma_k \hat{\theta}_k(s)$$

$\hat{\theta}_k(s)$ in $\mathscr{K}_s(\mathbb{I}_{0,\vartheta})$.(*Lu et al.*, 2010)

Implementation the distributed delay



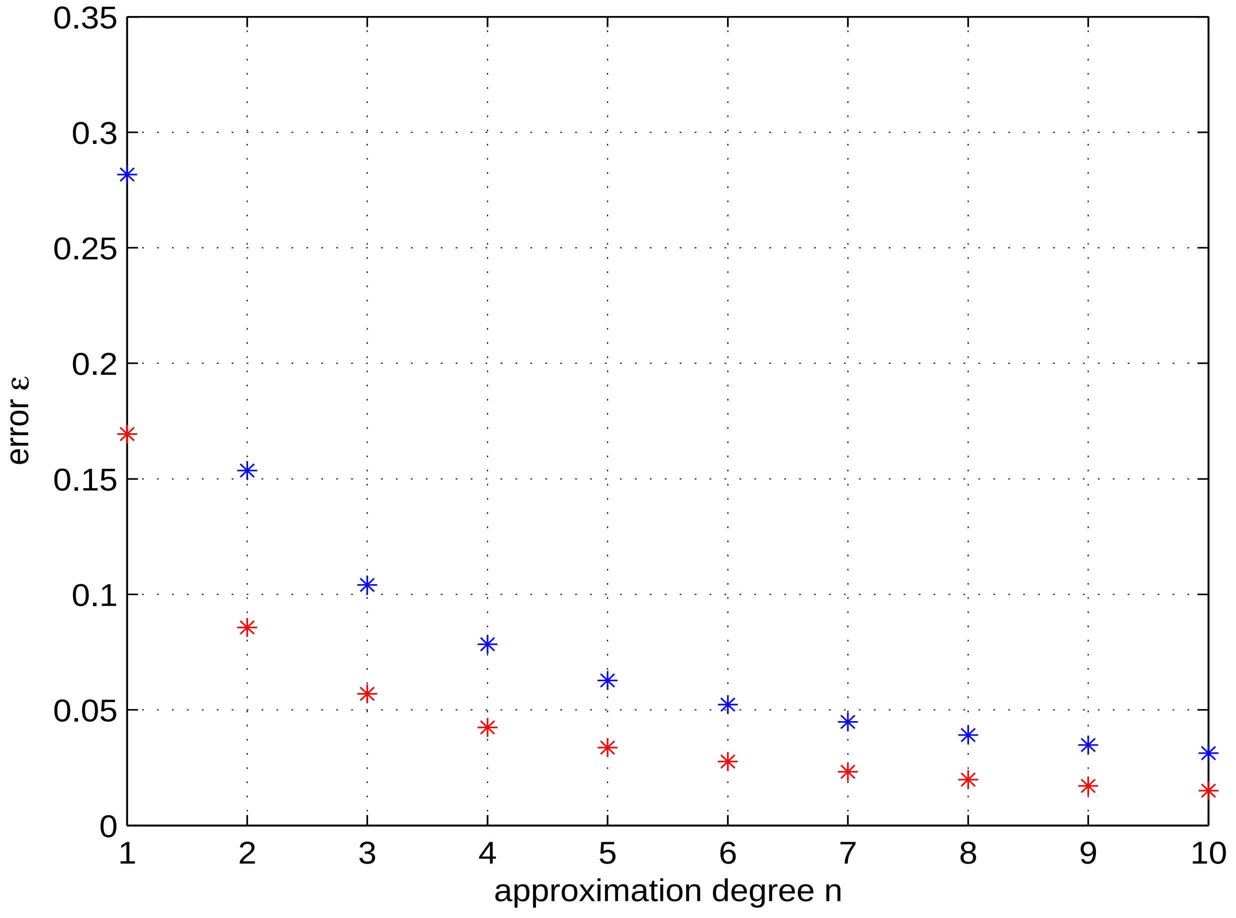
In time domain $\theta_1(t)$ and $\theta_{1,app}(t)$



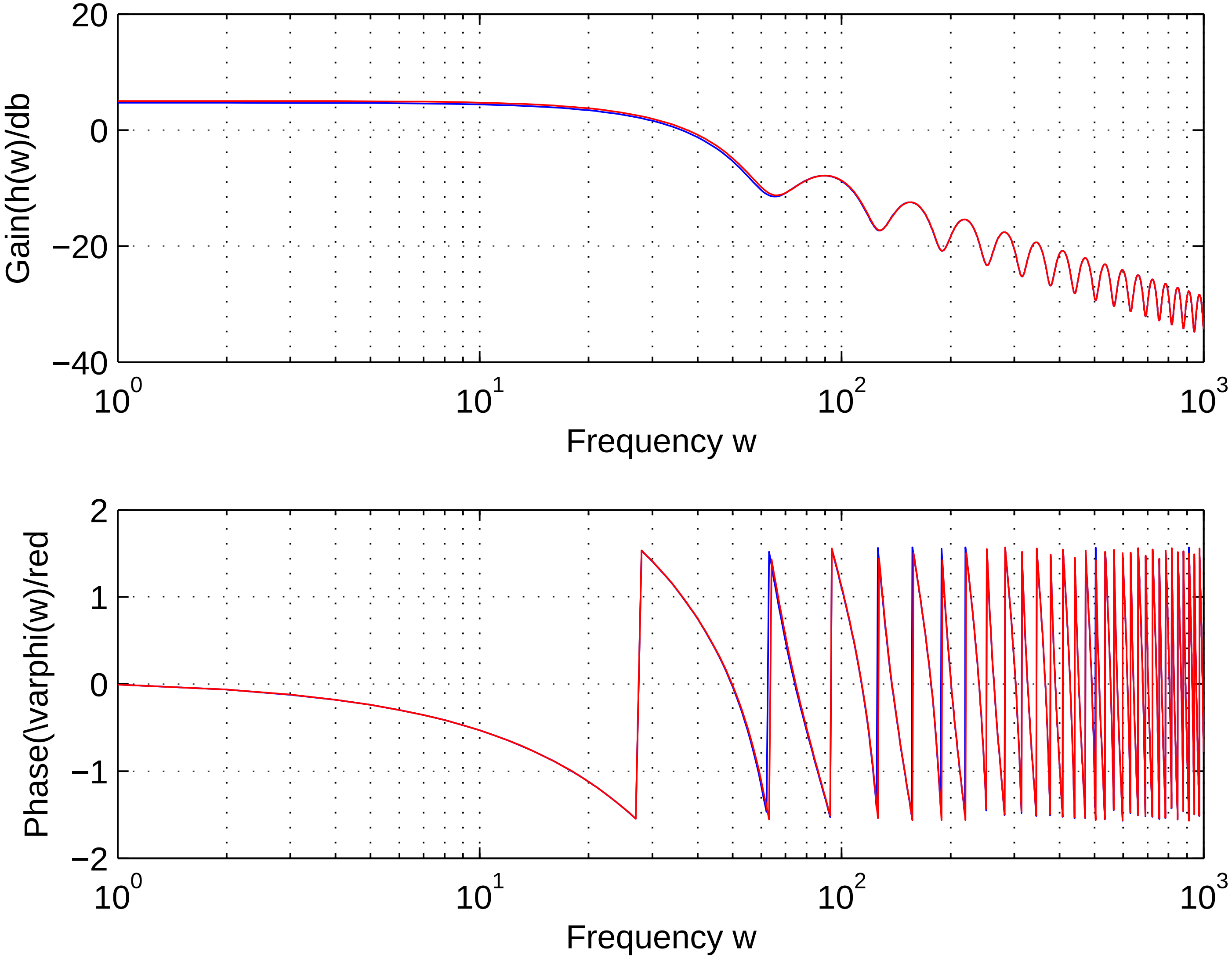
The relationship between error and approximation order ($\theta_{1,app}(t)$) :

$$t = -\alpha^{-1}\ln\rho$$

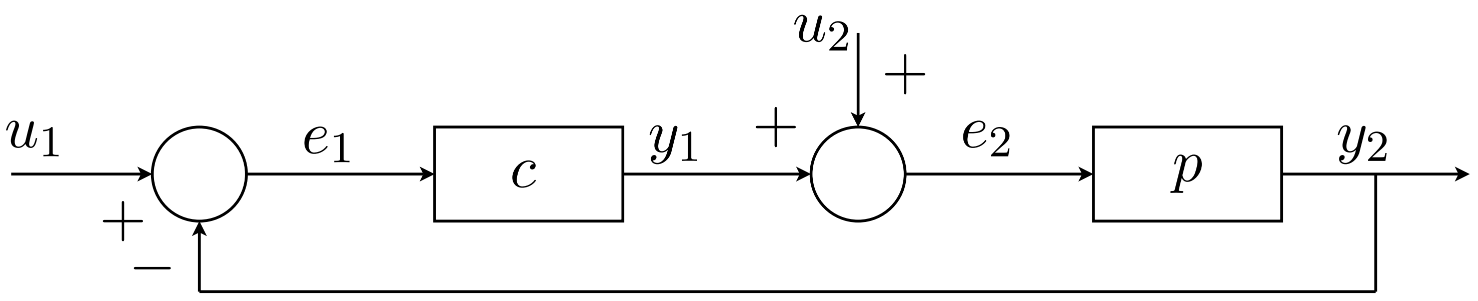
$$\theta_{\lambda,n}(t) = \frac{1}{(1-e^{-\alpha\vartheta})^n}\sum_{k=0}^{n} C_n^k \Phi_\lambda\left(\frac{k}{n}\right)(e^{-\alpha t}-e^{-\alpha\vartheta})^k(1-e^{-\alpha t})^{n-k}$$



Frquency properties $\theta_1(t)$ and $\theta_{1,app}(t)$:



Application for stabilization



$$p = \frac{n}{d}, c = \frac{n_c}{d_c}$$

The equations describing the closed-loop system are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = H(p,c) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where

$$H(p,c) = \begin{bmatrix} c/(1+pc) & -pc/(1+pc) \\ pc/(1+pc) & p/(1+pc) \end{bmatrix}$$

– Let p be a distribution.

– (n, d) : coprime factorization of p over $\mathscr{A}$

– Compensator c with coprime factorization (n_c, d_c)

Sufficient and necessary condition for stabilization :

The pair(p, c) is stable if and only if

$$\Phi = nn_c + dd_c$$

where Φ is a unit of $\mathscr{A}$.

Example

The plant

$$\hat{p} = \frac{\mathrm{e}^{-s}}{s-1}$$

A comprim factorization $\hat{n} = \frac{\mathrm{e}^{-s}}{s+1}, \hat{d} = \frac{s-1}{s+1}$

$$\hat{n} 2\mathrm{e} + \hat{d}\left(1 + 2\frac{1-\mathrm{e}^{-(s-1)}}{s-1}\right) = 1$$

A stabilizing compensator is $\hat{c}(s) = \frac{2\mathrm{e}}{1+2\frac{1-\mathrm{e}^{-(s-1)}}{s-1}}$. A realization is

$$u(t) = 2\mathrm{e}r(t) - 2\int_0^1 \mathrm{e}^{\tau} u(t-\tau)\mathrm{d}\tau - 2\mathrm{e}y(t)$$

Theorem

The plant of the system is $p(s) = \frac{\hat{n}}{\hat{d}}$, the compensator of the system is $\hat{c}(s) = \frac{\hat{n_c}}{\hat{d_c}}$. Using approximation for $\hat{n_c}(s), \hat{d}_c(s)$, the system is stable if

$$\max(\varepsilon_n,\ \varepsilon_d) < \left\| \begin{matrix} \hat{n} \\ \hat{d} \end{matrix} \right\|_{\hat{\mathscr{A}}}^{-1}$$

where $\varepsilon_{\hat{n}} = \hat{n}_{app} - \hat{n}$, $\varepsilon_{\hat{d}} = \hat{d}_{app} - \hat{d}$

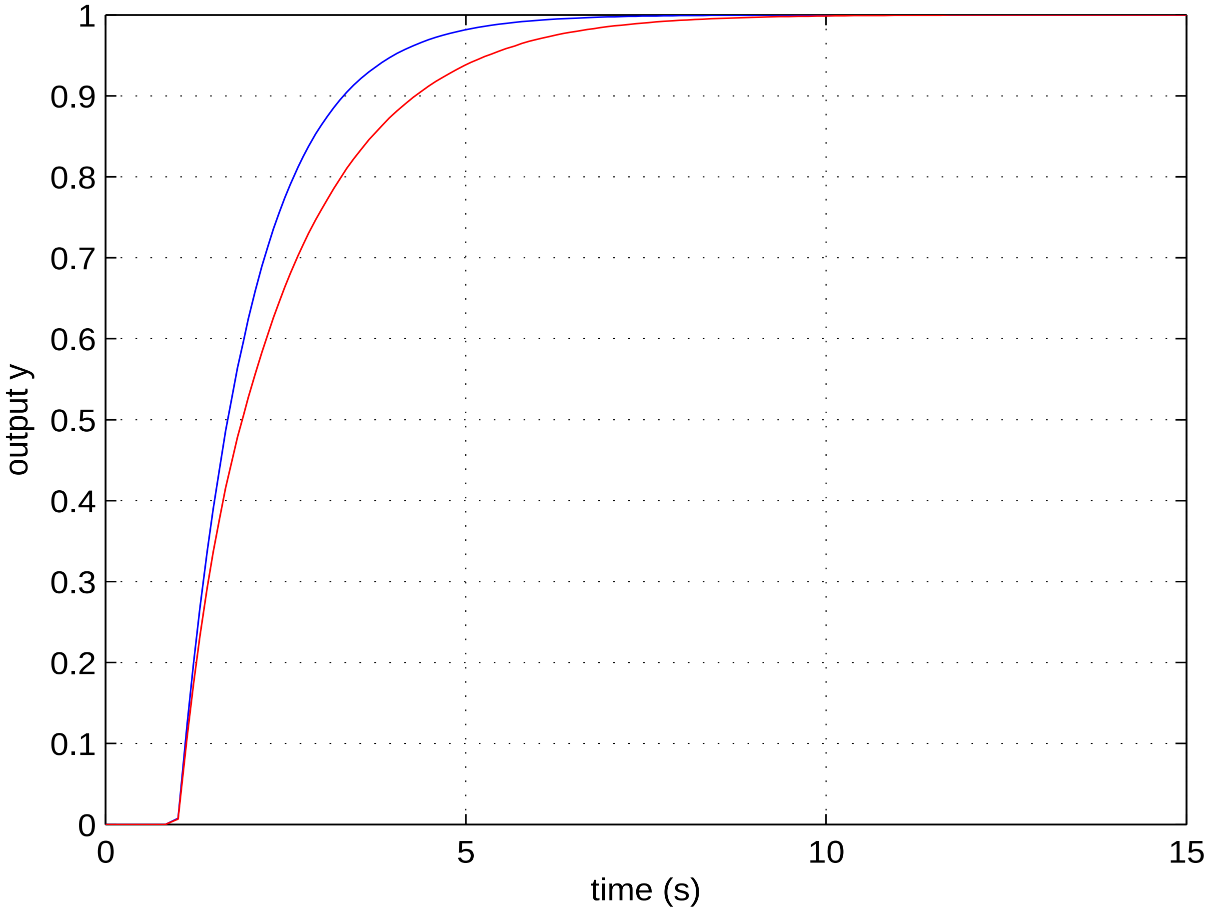
Stabilization example :

$$p(s) = \frac{\mathrm{e}^{-s}}{s-1}$$

a stabilizing compensator is $\hat{c}(s) = \frac{2\mathrm{e}}{1+2\frac{1-\mathrm{e}^{-(s-1)}}{s-1}}$.

- $\varepsilon_n = 0, \varepsilon_d = \varepsilon$
- $\hat{d} = \frac{s-1}{s+1}$
- $\varepsilon < \frac{1}{3}$

Time response :



Conclusion

Contribution

1. General methodology to realize an approximation for distributed delay
2. Approximation in $\mathscr{L}_1$
3. The kernel approximation
4. Graph topology over a general convolution algebra

Perspective

1. Minimize the order of the approximation
2. Effective calculation of the approximation
3. Other control problem, finite spectrum assignment
4. Generalization for distributed parameter systems